Pergamon

Topology Vol. 35, No. 4, pp. 969–1003, 1996

0040–9383/96/$15.00 + 0.00

0040–9383(95)00054–2

HYPERCOHOMOLOGY OF MILNOR FIBRES

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(*Received* 7 *September* 1993; *in revised form* 28 *March* 1995)

0. INTRODUCTION

IN THIS paper, we prove a number of results which help describe the hypercohomology of general Milnor fibres with coefficients in bounded, constructible complexes of sheaves. The principal goal of all of these results is to provide a means of algebraically calculating some pieces of data typically associated with a complex analytic singularity: the cohomology groups of the Milnor fibre of a function, the cohomology of the complex link of a space at a point, and the characteristic cycle of a complex of sheaves. Some of these results have appeared in substantially weaker forms in our earlier paper [35].

The results of this paper fall into three main categories:

If X is an analytic space, $\mathbf{F}^\bullet$ is a bounded, constructible complex of sheaves on X, and $f: X \to \mathbb{C}$ is a complex analytic function with a stratified isolated critical point at $x \in X$, then we will show in Section 3 that the relative hypercohomology, with coefficients in $\mathbf{F}^\bullet$, of a small ball around x modulo the Milnor fibre of f at x breaks up as a direct sum of powers of the hypercohomology of the normal data to each of the strata. With constant coefficients, this is closely related to a result of Lê in [23], Tibăr in [40], and, when X itself also has an isolated singular point at the origin, of Siersma in [38]; however, these three papers provide no method for determining how many times each stratum contributes to the direct sum. Our main result on this problem is the explicit calculation of the exponents that occur in this direct sum (Theorem 3.2). One implication of this calculation is that it allows one to specify algebraic data which implies the constancy of the stalk cohomology of the sheaf of vanishing cycles in a family of generalized isolated singularities (see Section 6 for a discussion of this and related problems).

Secondly, in Section 4, we examine the general case where the function $f: X \to \mathbb{C}$ has a stratified critical locus of arbitrary dimension; here, the description of the hypercohomology of the Milnor fibre becomes more difficult. We study the Milnor fibre of such an f by modding-out by a sufficiently generic hypersurface slice. We prove that, if $g: X \to \mathbb{C}$ is a second function which meets the hypersurface $V(f)$ "nicely", then the relative hypercohomology of the Milnor fibre of f modulo the Milnor fibre of $f|_{V(g)}$ breaks up as a direct sum of powers of the hypercohomology of the normal data to each of the strata of X (Theorem 4.2)—this is reminiscent of the isolated critical point case. Moreover, we can once again explicitly describe the exponents that occur in this direct sum.

As a corollary of our calculations, in Example 5.3, we recover and generalize the formula of Lê and Greuel (see [13, 17, 21] and [28, 5.11.a]) for the Milnor number of isolated complete intersections.

[†]Partially supported by National Science Foundation grant #DMS-9003498.

Finally, at the end of Section 4, we apply the results described above to calculate the cohomology of the complex link of a complex analytic space and, more generally, to obtain an inductive method for calculating the characteristic cycle of an arbitrary complex of sheaves (Corollary 4.6).

In example 5.6, we give a specific example of such a calculation: we start with a codimension 2 complete intersection $V(f, g)$ in $M := \mathbb{C}^4$, one which has a one-dimensional singular set, and then we calculate the characteristic cycles of the constant sheaf $\mathbb{C}^{\bullet}_{V(f,g)}$, the sheaf of vanishing cycles $\phi_g \mathbb{C}^{\bullet}_{V(f)}$, and of the iterated nearby and vanishing cycles $\phi_g \psi_f \mathbb{C}^{\bullet}_M$. The multiplicities occurring in these characteristic cycles provide important numerical information about the singularity; in particular, they generalize the data given by the Milnor number of an isolated singularity.

We need to use hypercohomology in this paper in order to deal with iterated nearby and vanishing cycles. The generality of using complexes of sheaves as coefficients actually simplifies some of the proofs, and in no instance does it lead to any added difficulty. Furthermore, using hypercohomology lets us deal with the important case of intersection cohomology coefficients.

1. BACKGROUND MATERIAL

In this section, we wish to go over the necessary background information on a number of topics, and we wish to fix some notation. Of course, even for those topics which we cover here, our treatment is necessarily cursory. References for analytic stratifications, including Whitney stratifications, are [12, 16, 27, 39]. References for good and Thom stratifications are [14, 33, 36]. References for the relative polar curve are [17, 18, 39]. References for the characteristic cycle are [6, 9, 16, 24, 30].

Unfortunately, in so short a paper, it is not possible to give any of the necessary background information on constructible complexes of sheaves, perverse sheaves, and the neighbouring and vanishing cycles. References for these topics include [2, 5, 7, 11, 16, 22].

Throughout this paper, we will be interested in the following situation: X will be a reduced s-dimensional complex analytic space contained in some open subset $\mathcal{U} \subseteq \mathbb{C}^N$ (we are *not* assuming that X is pure-dimensional). We are primarily interested in the germ of X at some point so, for convenience, we frequently assume that the origin is in X. We write ΣX for the singular set of X.

Let $f:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ be an analytic map (or, a map germ). We often write $V(f)$ for $f^{-1}(0)$. If X is a manifold, then we write Σf for the critical locus of f.

We recall a number of definitions.

Definition 1.1. A *(complex analytic) stratification* of X is a locally finite decomposition of X into complex analytic submanifolds (the *strata*) such that the closure of each stratum is complex analytic and a union of strata (the *condition of the frontier*). See [12, 36]. Unless we explicitly state otherwise, all stratifications in this paper are assumed to be complex analytic.

A *refinement* of a stratification $\mathcal{S}$ of X is a stratification $\mathcal{R}$ of X such that each stratum of $\mathcal{S}$ is a union of strata of $\mathcal{R}$.

A *Whitney stratification* is a stratification of X which satisfies Whitney's conditions A and B (again, see [12, 36]). When we use this term in this paper, we will also require that the strata are connected, so that the topology along the strata is not just locally constant, but actually constant.

For any stratification, $\mathscr{S}$, of X and for any $\mathbf{p} \in S_\beta \in \mathscr{S}$, we define a *degenerate tangent plane of $\mathscr{S}$ at* $\mathbf{p}$ to be an element, $\mathscr{T}$, of some Grassmanian which occurs as a limit of tangent planes from a higher-dimensional stratum at $\mathbf{p}$, i.e. $\mathscr{T} = \lim_{\mathbf{p}_i} T_{\mathbf{p}_i} S_\alpha$, where $S_\alpha \neq S_\beta$, $\mathbf{p}_i \in S_\alpha$, and $\mathbf{p}_i \to \mathbf{p}$ (see [12]). This notion is particularly interesting when the stratification satisfies Whitney's condition A, for then if $\mathbf{p} \in S_\beta \in \mathscr{S}$ and $\mathscr{T}$ is a degenerate tangent plane of $\mathscr{S}$ at $\mathbf{p}$, it follows that $T_\mathbf{p} S_\beta \subseteq \mathscr{T}$.

A *degenerate covector of $\mathscr{S}$ at a point* $\mathbf{p} \in X$ is a covector which vanishes on a degenerate tangent plane of $\mathscr{S}$ at $\mathbf{p}$, i.e. an element η of $T^*_\mathbf{p}\mathscr{U}$ such that there exists a degenerate tangent plane, $\mathscr{T}$, of $\mathscr{S}$ at $\mathbf{p}$ such that $\eta(\mathscr{T}) = 0$. Letting $T^*_{S_\alpha}\mathscr{U}$ denote the conormal bundle to S_α in $\mathscr{U}$, η being a degenerate covector of $\mathscr{S}$ at $\mathbf{p}$ is equivalent to $(\mathbf{p}, \eta)$ being in the closure of some $T^*_{S_\alpha}\mathscr{U}$ where $\mathbf{p} \notin S_\alpha$.

Given two strata, S_α and S_β, of a stratification of X, the pair (S_α, S_β) is said to satisfy the *Thom condition with respect to* f (or the a_f condition) at a point $\mathbf{p} \in S_\beta$ provided that the differential df has constant rank on S_α and, for any sequence of points $\mathbf{p}_i \in S_\alpha$ such that $\mathbf{p}_i \to \mathbf{p}$ and $\ker d_{\mathbf{p}_i} f|_{S_\alpha}$ converges to some $\mathscr{T}$ in the appropriate Grassmanian, we have $\ker d_\mathbf{p} f|_{S_\beta} \subseteq \mathscr{T}$ (see [12, 14, 36]). (S_α, S_β) satisfies the *Thom condition with respect to* f provided that df has constant rank on S_β and (S_α, S_β) satisfies the Thom condition with respect to f at each point $\mathbf{p} \in S_\beta$.

A *Thom stratification* of X with respect to f is a Whitney stratification of X in which each pair of strata satisfies the Thom condition. As our map f has codomain $\mathbb{C}$, the existence of Thom stratifications is guaranteed by [15].

Suppose that $\mathscr{C}$ is a locally finite collection of analytic subsets of X. Then, we say that a stratification of X is *adapted to* $\mathscr{C}$ provided that each element of $\mathscr{C}$ is a union of strata. In particular, to say that a stratification is adapted to the origin merely means that the origin is a stratum.

By [12, 1.7], given a stratification, $\mathscr{S}$, of X and a locally finite collection of analytic subsets, $\mathscr{C}$, one can refine $\mathscr{S}$ to obtain a Whitney stratification of X which is adapted to $\mathscr{C}$.

A *good stratification of X relative to* f is a stratification, $\mathscr{S}$, of X which is adapted to $V(f)$ such that $\{S_\alpha \in \mathscr{S} \mid S_\alpha \nsubseteq V(f)\}$ is a Whitney stratification of $X - V(f)$ and such that for any pair of strata (S_α, S_β) such that $S_\alpha \nsubseteq V(f)$ and $S_\beta \subseteq V(f)$, the Thom condition is satisfied. In our setting this is equivalent to: if $\mathbf{p} \in S_\beta \subseteq V(f)$ and $\mathbf{p}_i \in S_\alpha \nsubseteq V(f)$ are such that $\mathbf{p}_i \to \mathbf{p}$ and $T_{\mathbf{p}_i} V(f|_{S_\alpha} - f|_{S_\alpha}(\mathbf{p}_i))$ converges to some $\mathscr{T}$, then $T_\mathbf{p} S_\beta \subseteq \mathscr{T}$; note that this implies that the pair (S_α, S_β) must satisfy Whitney's condition A. In a good stratification, we call the strata which comprise $V(f)$ the *good strata*.

This notion of a good stratification is a generalization of the notion defined in [14]. Good stratifications always exist, since we may take a Thom stratification adapted to $V(f)$. Note that any refinement of a good stratification which refines only the good strata is again a good stratification.

Given any analytic stratification, $\mathscr{S} = \{S_\alpha\}$, of X, we define the following:

Definition 1.2. The *critical locus of f relative to* $\mathscr{S}$, $\Sigma_\mathscr{S} f$, is the union of the critical loci of f restricted to each of the strata, i.e. $\Sigma_\mathscr{S} f = \bigcup_\alpha \Sigma(f|_{S_\alpha})$. It is a trivial exercise to show that $\Sigma_\mathscr{S} f$ is closed if $\mathscr{S}$ satisfies Whitney's condition A.

A *critical point of f relative to* $\mathscr{S}$ is a point $\mathbf{p} \in \Sigma_\mathscr{S} f$. If the stratification $\mathscr{S}$ is clear, we refer to the elements of $\Sigma_\mathscr{S} f$ simply as *stratified critical points of* f. Naturally, we refer to the image under f of stratified critical points as *stratified critical values*.

If $\mathbf{p}$ is an isolated point of $\Sigma_\mathscr{S} f$, we call $\mathbf{p}$ a *stratified isolated critical point of* f (with respect to $\mathscr{S}$).

If $\mathscr{S}$ is a Whitney stratification of X and $f:(X, \mathbf{p}) \to (\mathbb{C}, 0)$ has a stratified isolated critical point at $\mathbf{p}$, then it is trivial to show that

$$\{S_\alpha - V(f), S_\alpha \cap V(f) - \mathbf{p}, \ \mathbf{p} \mid S_\alpha \in \mathscr{S}\}$$

is a good stratification for f in a neighbourhood of $\mathbf{p}$; we call this stratification the *good statification induced by f at* $\mathbf{p}$.

A critical point, $\mathbf{p}$, of f relative to $\mathscr{S}$ is a *decent critical point of f relative to $\mathscr{S}$* provided that f has analytic extension, $\tilde{f}$, to an open neighbourhood of $\mathbf{p}$ in $\mathscr{U}$ such that $d\tilde{f}_\mathbf{p}$ is not a degenerate covector to $\mathscr{S}$ at $\mathbf{p}$ (this is independent of the extension; see [12]). We call a critical point which is not decent *wicked.* (Of course, we would rather use the terms *non-degenerate* and *degenerate* in place of decent and wicked, but these terms are already taken. See below.) The point of this definition is that when one perturbs f slightly, a decent critical point, $\mathbf{p}$, of f relative to $\mathscr{S}$ will "split" into new critical points which are all contained in the same stratum as $\mathbf{p}$—that is, no new critical points will appear near $\mathbf{p}$ on larger-dimensional strata.

If $\mathscr{S}$ is a Whitney stratification, then a critical point $\mathbf{p} \in S \in \mathscr{S}$ of f relative to $\mathscr{S}$ is *non-degenerate* if and only if $\mathbf{p}$ is a decent critical point of f and $f|_S$ has a complex non-degenerate singularity at $\mathbf{p}$ (i.e. the complex Hessian of $f|_S$ is non-singular at $\mathbf{p}$—this condition is vacuous if $\mathbf{p}$ is a point-stratum). Non-degenerate critical points are necessarily isolated.

It is fundamental that

PROPOSITION 1.3. *Given an analytic map $f: X \to \mathbb{C}$, a stratification $\mathscr{S}$ of X, and a point $\mathbf{p} \in f^{-1}(0)$, there exists a neighbourhood of $\mathbf{p}$ in which $\Sigma_{\mathscr{S}} f \subseteq f^{-1}(0)$.*

Proof. This is an easy application of the analytic curve selection lemma (see [28, 37]). Suppose to the contrary that we had an infinite sequence of points

$$\mathbf{p}_i \in \Sigma_{\mathscr{S}} f - f^{-1}(0)$$

such that $\mathbf{p}_i \to \mathbf{p}$. As $\mathscr{S}$ is locally finite, there must be some stratum S_α which itself contains an infinite number of these points; so, assume that $\mathbf{p}_i \in S_\alpha$. Thus, $\mathbf{p}$ is in the closure of the semi-analytic set $\Sigma(f|_{S_\alpha}) - f^{-1}(0)$.

Hence, by the curve selection lemma, there exists a real analytic curve, $\mathbf{c}(u)$ such that $\mathbf{c}(0) = \mathbf{p}$ and $\mathbf{c}(u) \in \Sigma(f|_{S_\alpha}) - f^{-1}(0)$ for $u \neq 0$, i.e. $d(f|_{S_\alpha})(\mathbf{c}(u))$ is identically zero. Thus, $f(\mathbf{c}(u))$ is a constant, and this constant must be zero since $\mathbf{c}(0) = \mathbf{p}$ and $f(\mathbf{p}) = 0$. Therefore, $f(\mathbf{c}(u))$ is identically zero—a contradiction as $\mathbf{c}(u) \notin f^{-1}(0)$ for $u \neq 0$. □

Definition 1.4. Given an analytic map $f: X \to \mathbb{C}$ and a point $\mathbf{p} \in f^{-1}(0)$, the *Milnor fiber of f at* $\mathbf{p}$, $F_{f,\mathbf{p}}$, is defined to be the (homeomorphism-type of the) space obtained by taking a small open ball, $\mathring{B}_\varepsilon(\mathbf{p})$, in $\mathbb{C}^N$, centred at $\mathbf{p}$, intersecting with X, and intersecting with $f^{-1}(\xi)$, where $0 < |\xi| \ll \varepsilon$, i.e.

$$F_{f,\mathbf{p}} := \mathring{B}_\varepsilon(\mathbf{p}) \cap X \cap f^{-1}(\xi).$$

This is independent of all choices made. If we are only interested in the homotopy-type of the Milnor fibre, we may use the closed ball in the definition (See [10, 12, 20]).

By Proposition 1.3, if X has a given stratification $\mathscr{S} = \{S_\alpha\}$, then the Milnor fibre has an *induced stratification* given by $F_{f,\mathbf{p}} \cap \mathscr{S} := \{F_{f,\mathbf{p}} \cap S_\alpha\}$. Moreover, if the strata of $\mathscr{S}$ which are not contained in $V(f)$ satisfy the Whitney conditions, then this induced stratification on the Milnor fibre is a Whitney stratification; in particular, this will be the case if $\mathscr{S}$ is a good stratification of X relative to f.

If $\mathbf{p} \in X$ and L is a generic linear form, then the Milnor fibre of $L - L(\mathbf{p})$ at $\mathbf{p}$ is the *complex link* of X at $\mathbf{p}$, which we denote by $\mathbb{L}_{X,\mathbf{p}}$. The homotopy-type of this space is an analytic invariant of the germ $(X, \mathbf{p})$ (see [26]). By [12], if we take any Whitney stratification, $\mathscr{S}$, of X and take any analytic function $f: X \to \mathbb{C}$ which has $\mathbf{p}$ as a non-degenerate critical point relative to $\mathscr{S}$, then the Milnor fibre of $f - f(\mathbf{p})$ has the homotypy-type of the complex link.

We must now give a number of definitions from [12]. We give these definitions only in our complex analytic setting; this is the setting of Part II of [12].

Definition 1.5. Let $\mathscr{S}$ be a Whitney stratification of $X \subseteq \mathscr{U}$. (Recall that $\mathscr{U}$ is an open subset of some affine space.)

Let $\mathbf{p} \in X$ be a decent, stratified isolated critical point of an analytic map $f: X \to \mathbb{C}$. Then, for all non-zero complex w, $\mathbf{p}$ is a non-depraved critical point of the real analytic function $|f - f(\mathbf{p}) - w|^2$ in the sense of [12], and has well-defined local Morse data which is independent of w; up to homotopy, this local Morse data is given by the pair

$$(B_\varepsilon(\mathbf{p}), B_\varepsilon(\mathbf{p}) \cap f^{-1}(v)) \simeq (\mathring{B}_\varepsilon(\mathbf{p}), (\mathring{B}_\varepsilon(\mathbf{p}) \cap f^{-1}(v))$$

where $0 < |v - f(\mathbf{p})| \ll \varepsilon \ll 1$ and $\simeq$ denotes homotopy-equivalence. We define the *local Morse data of f at* $\mathbf{p}$ to be the homotopy-type of the pair $(B_\varepsilon(\mathbf{p}), B_\varepsilon(\mathbf{p}) \cap f^{-1}(v))$. This is the same homotopy-type as the local Morse data at $\mathbf{p}$ of the real analytic function $\mathrm{Re} f$.

Let $\mathbf{p} \in S_\alpha \in \mathscr{S}$. Let N' be a complex analytic subset of $\mathscr{U}$ which transversely intersects S_α in the single point $\mathbf{p}$. Then, $N := N' \cap X$ is a *normal slice of X at* $\mathbf{p}$ and, in a neighbourhood of $\mathbf{p}$, there is an induced Whitney stratification of N given by

$$\{S_\alpha \cap N' \mid S_\alpha \in \mathscr{S}\};$$

note that $\mathbf{p}$ will be a point-stratum in this induced stratification.

Let $S_\alpha \in \mathscr{S}$. For any $\mathbf{p} \in S_\alpha$ and for any normal slice, N, to X at $\mathbf{p}$, we define the (*complex*) *link* of S_α to be the stratified homeomorphism-type of the complex link of N at $\mathbf{p}$; this is independent of all choices made. We denote the link of S_α by either $\mathbb{L}_{S_\alpha}$ or, simply, $\mathbb{L}_\alpha$. Bear in mind that this means that we now have two notations for the complex link at a point: $\mathbb{L}_{X,\mathbf{p}}$ when no stratification has been specified, and $\mathbb{L}_{\{\mathbf{p}\}}$ when we have a fixed stratification in which we assume that $\mathbf{p}$ is a point-stratum.

Note that if S_α is a top-dimensional stratum of X, then a normal slice to S_α will locally consist of a single point and, hence, $\mathbb{L}_\alpha$ will be empty.

We define the *normal data to $S_\alpha \in \mathscr{S}$* to be the local Morse data at $\mathbf{p} \in S_\alpha$ of any complex analytic function germ on a normal slice to S_α at $\mathbf{p}$, $f: N \to \mathbb{C}$, which has $\mathbf{p}$ as a decent, stratified isolated critical point. Again, this is independent of all the choices made. By Corollary 1 of [12, II.2.4], the normal data to S_α is given by taking the cone on the link of S_α modulo the link of S_α, i.e. the normal data is given by $(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha)$. (Here, by convention, we set the cone on the empty set to be a single point.)

Definition 1.6. Suppose that we are given two maps $f: X \to \mathbb{C}$ and $g: X \to \mathbb{C}$. Define the map $\Phi := (f, g): X \to \mathbb{C}^2$. If Y is an analytic subset of X, we define $\Gamma_{f,g}(Y)$ to be the closure in X of the critical locus of $\Phi|_{Y - \Sigma Y - V(f)}$. This is called the *relative polar variety of Y with respect to f and g.*

This is a mild generalization of the notion defined by Lê and Teissier; see [18, 26, 39]. Normally, one makes this definition only if g is linear or, after an analytic change of coordinates, if g does not have a critical point at the origin. We will concentrate most of our attention on the case where g has a stratified isolated singularity. Note the asymmetry in this definition: $\Gamma_{f,g}(Y)$ is allowed to have components in $V(g)$, but not in $V(f)$.

Given a stratification, $\mathscr{S}$, of X, for each $S_\alpha \in \mathscr{S}$, $\Gamma_{f,g}(S_\alpha)$ is thus the closure of $\operatorname{crit}(\Phi|_{S_\alpha - V(f)})$. The union $\bigcup_\alpha \Gamma_{f,g}(S_\alpha)$ is called *the relative polar variety of f and g with respect to $\mathscr{S}$*, and we denote it by $\Gamma_{f,g}(\mathscr{S})$ (or, simply, $\Gamma_{f,g}$ if the stratification is clear); see [19]. Note that if $S_\alpha \subseteq V(f)$, then $\Gamma_{f,g}(S_\alpha) = \emptyset$.

We define the *symmetric relative polar variety of Y with respect to f and g*, $\tilde{\Gamma}_{f,g}(Y)$, to be the closure in X of the critical locus of $\Phi|_{Y - \Sigma Y - V(f) - V(g)}$. This is easily seen to be equal to the closure in X of $\Gamma_{f,g}(Y) - V(g)$, i.e. the union of the components of $\Gamma_{f,g}(Y)$ which are not contained in $V(g)$. Naturally, we also define *the symmetric relative polar variety of f and g with respect to $\mathscr{S}$*, $\tilde{\Gamma}_{f,g}(\mathscr{S})$, to be the union $\bigcup_\alpha \tilde{\Gamma}_{f,g}(S_\alpha)$. We use the term "symmetric" since we obviously have $\tilde{\Gamma}_{f,g}(Y) = \tilde{\Gamma}_{g,f}(Y)$.

In the special case where f and g are such that $\Gamma_{f,g}(\mathscr{S})$ (resp. $\tilde{\Gamma}_{f,g}(\mathscr{S})$ is one-dimensional, we naturally refer to the (resp. symmetric) polar variety as the *(resp. symmetric) polar curve* and emphasize the fact that it is one-dimensional by writing $\Gamma^1_{f,g}(\mathscr{S})$ (resp. $\tilde{\Gamma}^1_{f,g}(\mathscr{S})$). In this case, we also wish to give the (resp. symmetric) polar curve the structure of a cycle (actually, a cycle germ at the origin), so we must attach some multiplicity to each component of this curve.

To do this, for each (one-dimensional) component, v, of $\Gamma^1_{f,g}(\mathscr{S})$ (resp. $\tilde{\Gamma}^1_{f,g}(\mathscr{S})$), let S_v denote the stratum which contains $v - \mathbf{0}$ near the origin. If S_v is itself one-dimensional, we assign the multiplicity 1 to v (that is, we consider v with its reduced structure). Now, to each v for which S_v is not one-dimensional, we assign the multiplicity given by the Milnor number of the map g restricted to $S_v \cap V(f - f(\mathbf{p}))$ at any point $\mathbf{p} \in v - \mathbf{0}$ sufficiently close to the origin. Note that $S_v \cap V(f - f(\mathbf{p}))$ is a manifold at $\mathbf{p}$ by Proposition 1.3, and that g restricted to this set has an isolated critical point at $\mathbf{p}$ since v is one-dimensional.

As it will always be clear whether we are considering the (resp. symmetric) polar curve as a set or a cycle, we introduce no new notation for this cycle.

Remark 1.7. In the above definition of the cycle structure on the symmetric polar curve, to each component, v, of the symmetric polar curve for which S_v is not one-dimensional, we assign the multiplicity given by the Milnor number of the map g restricted to $S_v \cap V(f - f(\mathbf{p}))$ at any point $\mathbf{p} \in v - \mathbf{0}$ sufficiently close to the origin. Because both $V(g - g(\mathbf{p}))$ and $V(f - f(\mathbf{p}))$ transversely intersect $\tilde{\Gamma}^1_{f,g}$ at such a point $\mathbf{p}$, it is easy to show that this Milnor number $\mu_{\mathbf{p}}(g|_{S_v \cap V(f - f(\mathbf{p}))})$ is also equal to $\mu_{\mathbf{p}}(f|_{S_v \cap V(g - g(\mathbf{p}))})$. This number is also the multiplicity at $\mathbf{p}$ of the subscheme of S_v given in the following manner: let $u_0, \ldots, u_k$ be local coordinates for S_v at $\mathbf{p}$. Suppose $\partial f/\partial u_0 \neq 0$ at $\mathbf{p}$. Then, $\partial g/\partial u_0 \neq 0$ at $\mathbf{p}$ and $\tilde{\Gamma}^1_{f,g}$ is defined as a subscheme of S_v at $\mathbf{p}$ by

$$V\left(\frac{\partial f}{\partial u_0}\frac{\partial g}{\partial u_1} - \frac{\partial f}{\partial u_1}\frac{\partial g}{\partial u_0}, \ldots, \frac{\partial f}{\partial u_0}\frac{\partial g}{\partial u_k} - \frac{\partial f}{\partial u_k}\frac{\partial g}{\partial u_0}\right).$$

Without assuming that some particular partial derivative is non-zero, this subscheme is given by

$$V\left(\frac{\partial(f,g)}{\partial(u_{j_1}, u_{j_2})} : 0 \leqslant j_1 < j_2 \leqslant k\right)$$

where $\partial(f, g)/\partial(u_{j_1}, u_{j_2})$ denotes

$$\frac{\partial f}{\partial u_{j_1}}\frac{\partial g}{\partial u_{j_2}} - \frac{\partial f}{\partial u_{j_2}}\frac{\partial g}{\partial u_{j_1}},$$

a 2×2 minor of the Jacobian matrix of the map (f, g).

We can remove the reference to the local coordinates. Let S_v be a $k+1$-dimensional stratum which is defined, with its reduced structure, at $\mathbf{p}$ by the vanishing of $h_1, \ldots, h_l$ in $\mathbb{C}^N$. Let $J_i(h_1, \ldots, h_l, f, g)$ denote the ideal in $\mathcal{O}_{\mathbf{p}}^N$ generated by the $i\times i$ minors of the Jacobian matrix of the map $(h_1, \ldots, h_l, f, g)$. Then, the above subscheme is defined as a subscheme of $\mathbb{C}^N$ by

$$V(h_1, \ldots, h_l, J_{N-k+1}(h_1, \ldots, h_l, f, g))$$

i.e. the subscheme of $\mathcal{O}_{\mathbf{p}}^N$ defined at $\mathbf{p}$ by the vanishing of $h_1, \ldots, h_l$ and the $(N-k+1)\times(N-k+1)$ minors of the Jacobian matrix of the map $(h_1, \ldots, h_l, f, g)$.

This is the *critical space* of the map $(f, g)|_{S_v}$ at $\mathbf{p}$, and it is independent of the choice of the defining equations h_i; see [28, 4.A]. It is important to note that we are *not* assuming that $h_1, \ldots, h_l$ are a minimal set of defining functions for S_v at $\mathbf{p}$.

Example 1.8. Consider, for example, $X := V(h) \subseteq \mathbb{C}^3$, where $h = x^2 + y^2 + z^2$, and let $f = x^3$ and $g = y$. Then, at points not contained in $V(f)$ or $V(g)$, $\tilde{\Gamma}^1_{f,g}(X)$ is defined by the vanishing of h and the determinant of the matrix

$$\begin{pmatrix} 2x & 2y & 2z \\ 3x^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So, $\tilde{\Gamma}^1_{f,g}(X)$ is defined by starting with the cycle

$$V(6x^2z, x^2+y^2+z^2) + 2V(x, y^2+z^2) + V(z, x^2+y^2)$$

and discarding those components which are contained in $V(f)$ or $V(g)$. Hence, as cycles,

$$\tilde{\Gamma}^1_{f,g}(X) = V(z, x^2+y^2).$$

Example 1.9. Let us now consider the above example with a slightly altered f. Let $X = V(h) \subseteq \mathbb{C}^3$, where $h = x^2 + y^2 + z^2$, and let $f = x^3 + yz$ and $g = y$. Then, at points not contained in $V(f)$ or $V(g)$, $\tilde{\Gamma}^1_{f,g}(X)$ is defined by the vanishing of h and the determinant of the matrix

$$\begin{pmatrix} 2x & 2y & 2z \\ 3x^2 & z & y \\ 0 & 1 & 0 \end{pmatrix}.$$

So, $\tilde{\Gamma}^1_{f,g}(X)$ is defined by starting with the cycle

$$V(6x^2z - 2xy, x^2+y^2+z^2) = V(x, y^2+z^2) + V(3xz - y, x^2+y^2+z^2)$$

and discarding those components which are contained in $V(f)$ or $V(g)$; however, a quick check reveals that there are no components contained in $V(f)$ or $V(g)$. Hence, as cycles,

$$\tilde{\Gamma}^1_{f,g}(X) = V(x, y^2+z^2) + V(3xz - y, x^2+y^2+z^2).$$

We now wish to give some conditions under which the polar variety of f and g is one-dimensional.

Definition 1.10. Let $\mathscr{S}$ be a good stratification of X relative to a function $f:(X, \mathbf{0}) \to (\mathbb{C}, 0)$. We say that $g:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ is *prepolar with respect to* $\mathscr{S}$ at the origin if and only if the origin is a stratified isolated critical point, i.e. $\mathbf{0}$ is an isolated point of $\Sigma_{\mathscr{S}} g$.

By Proposition 1.3, this is equivalent to: for any analytic extension, $\tilde{g}$, of g to an open neighbourhood of the origin in $\mathbb{C}^N$, $X \cap \Sigma\tilde{g}$ is empty or has the origin as an isolated point and $V(\tilde{g})$ transversely intersects each stratum of $\mathscr{S}$ in a neighbourhood of the origin, except perhaps at the origin itself.

Note that for a fixed good stratification, $\mathscr{S}$, for f, the collection of linear forms which are prepolar with respect to $\mathscr{S}$ will be generic.

We say simply that g is prepolar with respect to f provided that there exists a good stratification of X relative to f with respect to which g is prepolar. Note that this definition is extremely anti-symmetric—while $V(g)$ meets $V(f)$ in a nice way, $V(f)$ may have arbitrarily bad singularities when restricted to $V(g)$.

A function $g:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ is *tractable at the origin with respect to a good stratification* $\mathscr{S}$ *of* X *relative to* $f:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ if and only if $\dim_{\mathbf{0}} \tilde{\Gamma}^1_{f,g}(\mathscr{S}) \leqslant 1$ and, for all good strata S_α (i.e. S_α such that $S_\alpha \in \mathscr{S}$ and $S_\alpha \subseteq V(f)$), $g|_{S_\alpha}$ has no critical points in a neighbourhood of the origin except perhaps at the origin itself.

We say simply that $g:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ is *tractable at the origin with respect to* f if there exists a good stratification $\mathscr{S}$ for X such that g is tractable with respect to $\mathscr{S}$.

We wish to show that prepolarity implies tractability; first we need:

LEMMA 1.11. *Suppose that* $\mathscr{S}$ *is a good stratification of* X *for* f *at the origin, and that* g *is prepolar with respect to* $\mathscr{S}$ *at the origin. Then, in a neighbourhood of the origin, for all* $S_\alpha \in \mathscr{S}$,

$$\Gamma_{f,g}(S_\alpha) - V(f) \subseteq \bigcup_{S_\beta \subseteq \overline{S_\alpha}} \operatorname{crit}(f, g)|_{S_\beta}.$$

Proof. Let $\mathbf{p} \in \Gamma_{f,g}(S_\alpha) - V(f)$, and let $S_\beta \subseteq \overline{S_\alpha}$ denote the stratum containing $\mathbf{p}$. Then, there exists a sequence $\mathbf{p}_i \to \mathbf{p}$ with $\mathbf{p}_i \in \operatorname{crit}(f, g)|_{S_\alpha} - V(f)$. We wish to show that this implies that $\mathbf{p} \in \operatorname{crit}(f, g)_{|S_\beta}$.

Let $\tilde{f}$ and $\tilde{g}$ denote extensions of f and g, respectively, to an open neighbourhood of $\mathbf{p}$ in $\mathscr{U}$. By Proposition 1.3, $T_{\mathbf{p}} V(f|_{S_\beta} - f|_{S_\beta}(\mathbf{p}))$ exists and is equal to $T_{\mathbf{p}} V(\tilde{f} - \tilde{f}(\mathbf{p})) \cap T_{\mathbf{p}} S_\beta$; we will show that this is contained in $T_{\mathbf{p}} V(\tilde{g} - \tilde{g}(\mathbf{p}))$.

Since $\mathbf{p}_i \notin V(f)$, it follows from Proposition 1.3 that $T_{\mathbf{p}_i} V(\tilde{f} - \tilde{f}(\mathbf{p}_i))$ exists (for i sufficiently large), and that

$$T_{\mathbf{p}_i} V(f|_{S_\alpha} - f|_{S_\alpha}(\mathbf{p}_i)) = T_{\mathbf{p}_i} V(\tilde{f} - \tilde{f}(\mathbf{p}_i)) \cap T_{\mathbf{p}_i} S_\alpha.$$

Since $\mathbf{p} \notin V(f)$, $T_{\mathbf{p}_i} V(\tilde{f} - \tilde{f}(\mathbf{p}_i)) \to T_{\mathbf{p}} V(\tilde{f} - \tilde{f}(\mathbf{p}))$. Using Proposition 1.3 again and the fact that g is prepolar, $T_{\mathbf{p}_i} V(\tilde{g} - \tilde{g}(\mathbf{p}_i))$ exists and approaches $T_{\mathbf{p}} V(\tilde{g} - \tilde{g}(\mathbf{p}))$. As $\mathbf{p}_i \in \operatorname{crit}(f, g)|_{S_\alpha}$,

$$T_{\mathbf{p}_i} V(\tilde{f} - \tilde{f}(\mathbf{p}_i)) \cap T_{\mathbf{p}_i} S_\alpha \subseteq T_{\mathbf{p}} V(g - g(\mathbf{p})).$$

By taking a subsequence, if necessary, we may assume that $T_{\mathbf{p}_i} S_\alpha$ approaches some limit, say $\mathscr{T}$. As $\mathscr{S}$ is a good stratification, the strata that comprise $X - V(f)$ satisfy Whitney's condition A, and so $T_{\mathbf{p}}S_\beta \subseteq \mathscr{T}$. Therefore,

$$T_{\mathbf{p}}V(\tilde{f}-\tilde{f}(\mathbf{p}))\cap T_{\mathbf{p}}S_\beta \subseteq T_{\mathbf{p}}V(\tilde{g}-\tilde{g}(\mathbf{p}))$$

and we are finished. □

Proposition 1.12. *Suppose that $\mathscr{S}$ is a good stratification of X for f at the origin, and that g is prepolar with respect to $\mathscr{S}$ at the origin. Then, $\dim_{\mathbf{0}} V(g)\cap\Gamma_{f,g}(\mathscr{S}) \leqslant 0$ (where <0 indicates an empty germ at the origin). Hence, $\Gamma_{f,g}(\mathscr{S}) = \tilde{\Gamma}_{f,g}(\mathscr{S})$ and each of these sets is either one-dimensional or empty at the origin. In particular, g is tractable with respect to $\mathscr{S}$ relative to f.*

Proof. Let $S_\alpha\in\mathscr{S}$. We will first show that $V(g)\cap\Gamma_{f,g}(S_\alpha)\subseteq V(f)$ near the origin. Suppose not. Then, $\mathbf{0}$ is in the closure of $V(g)\cap\Gamma_{f,g}(S_\alpha) - V(f)$. Applying Lemma 1.11 and the curve selection lemma, there would exist a stratum $S_\beta\subseteq\overline{S_\alpha}$ and a real analytic curve $\mathbf{c}(u)$ with $\mathbf{c}(0)=\mathbf{0}$ and

$$\mathbf{c}(u)\in V(g)\cap \operatorname{crit}(f,g)|_{S_\beta} - V(f)$$

for $u\neq 0$. As $\mathbf{c}(u)\notin V(f)$, Proposition 1.3 tells us that $\mathbf{c}(u)\notin\Sigma(f|_{S_\beta})$. Also, as g is prepolar, $\mathbf{c}(u)\notin\Sigma(g|_{S_\beta})$. Therefore, since $\mathbf{c}(u)\in\operatorname{crit}(f,g)|_{S_\beta}$, along $\mathbf{c}(u)$, $dg|_{S_\beta}$ must be a non-zero scalar multiple of $df|_{S_\beta}$. But, $g(\mathbf{c}(u))\equiv 0$ and so, applying the chain rule, we find that $f(\mathbf{c}(u))\equiv 0$; a contradiction. Thus, $V(g)\cap\Gamma_{f,g}(S_\alpha)\subseteq V(f)$.

Now, we shall prove the proposition. Suppose $\dim_{\mathbf{0}} V(g)\cap\Gamma_{f,g}(\mathscr{S})\geqslant 1$. Let $\tilde{g}$ denote an extension of g to an open neighbourhood of the origin in $\mathbb{C}^N$. Let $\mathbf{p}\in V(g)\cap\Gamma_{f,g}(S_\alpha) - \mathbf{0}$ be close to the origin. By the above, $\mathbf{p}\in V(f)$ and so must be contained in some stratum $S_\beta\subseteq V(f)$. As g is prepolar, $\mathfrak{T} := T_{\mathbf{p}}V(\tilde{g})$ exists and $\mathfrak{T} + T_{\mathbf{p}}S_\beta = \mathbb{C}^N$.

As $\mathbf{p}\in\Gamma_{f,g}(S_\alpha)\cap V(f)$ and since $\overline{\Gamma_{f,g}(S_\alpha) - V(f)} = \Gamma_{f,g}(S_\alpha)$, there exists a sequence of points $\mathbf{p}_i\in\Gamma_{f,g}(S_\alpha) - V(f)$ such that $\mathbf{p}_i\to\mathbf{p}$. By taking a subsequence, we may assume that all $\mathbf{p}_i$ are contained in a single stratum $S_\gamma\subseteq\overline{S_\alpha}$, that $T_{\mathbf{p}_i}V(f|_{S_\gamma} - f(\mathbf{p}_i))\to\mathscr{T}$ (we may assume this by Proposition 1.3), and that $T_{\mathbf{p}_i}V(\tilde{g}-\tilde{g}(\mathbf{p}_i))\to\mathfrak{T}$.

By the good condition, $T_{\mathbf{p}}S_\beta\subseteq\mathscr{T}$. However, as $\mathbf{p}_i\in\Gamma_{f,g}(S_\alpha) - V(f)$, by Lemma 1.11

$$T_{\mathbf{p}_i}V(f|_{S_\gamma} - f(\mathbf{p}_i)) = T_{\mathbf{p}_i}V(\tilde{g}-\tilde{g}(\mathbf{p}_i))\cap T_{\mathbf{p}_i}S_\gamma$$

and so $\mathscr{T}\subseteq\mathfrak{T}$. This is a contradiction, since $T_{\mathbf{p}}S_\beta\subseteq\mathscr{T}\subseteq\mathfrak{T}$ and $\mathfrak{T} + T_{\mathbf{p}}S_\beta = \mathbb{C}^N$. □

Note that this implies that, given f and a good stratification $\mathscr{S}$, there is a generic set of linear forms, L, such that L is tractable with respect to $\mathscr{S}$ relative to f.

We now need to define a relative non-degeneracy condition between two functions. While the condition that we define is not particularly easy to verify in general, it is trivially satisfied in the case where the space X has an isolated singularity.

Definition 1.13. For any analytic stratification, $\mathscr{S}$, of X and any $f:(X,\mathbf{0})\to(\mathbb{C},0)$, we say that g is *decent* with respect to $\mathscr{S}$ relative to f provided that all critical points of g restricted to $F_{f,\mathbf{0}} - V(g)$ are decent with respect to the induced stratification on $F_{f,\mathbf{0}}$ (more precisely, if there exists an open neighbourhood, $\mathscr{U}$, of $\mathbf{0}$ such that, for all ξ sufficiently small, $\neq 0$, g has no wicked critical points when restricted to $\mathscr{U}\cap X\cap f^{-1}(\xi) - V(g)$ with its induced stratification). Note that this condition is automatic if $\mathscr{S}$ has only one stratum not contained in $V(f)\cup V(g)$.

PROPOSITION 1.14. *Let $\mathcal{S}$ be a good stratification of X for f at the origin. Then, for a generic choice of linear forms, L, L is decent with respect to $\mathcal{S}$ relative to f and, moreover, f is decent with respect to $\mathcal{S}$ relative L.*

Proof. We shall prove simultaneously that both conditions hold; we proceed a stratum at a time.

Let S_α be a stratum not contained in $V(f)$. Let DS_α be the subset of $S_\alpha \times G_{N-1}(\mathbb{C}^N)$ defined by

$$DS_\alpha = \{(\mathbf{p}, H) \in S_\alpha \times G_{N-1}(\mathbb{C}^N) \mid H = \text{the kernel of a degenerate covector at } \mathbf{p}\}.$$

By [12, 1.8], the dimension of $DS_\alpha \leqslant N-2$.

Let $\tilde{f}$ be an extension of f to an open neighbourhood of the origin in $\mathbb{C}^N$. By Proposition 1.3, we may define a map $\psi : DS_\alpha \to S_\alpha \times G_{N-2}(\mathbb{C}^N)$ by

$$\psi(\mathbf{p}, H) = (\mathbf{p}, H \cap T_{\mathbf{p}}\tilde{f}^{-1}\tilde{f}(\mathbf{p})).$$

Then, $\dim \overline{\operatorname{im} \psi} \leqslant N-2$, and so the fibre over the origin, $(\overline{\operatorname{im} \psi})_0$, has dimension $\leqslant N-3$.

We now wish to show that, for a generic linear form L, the kernel of L does not contain any element of $(\overline{\operatorname{im} \psi})_0$; for such an L clearly satisfies both non-degeneracy conditions. Let

$$W := \{(K, H) \in (\overline{\operatorname{im} \psi})_0 \times G_{N-1}(\mathbb{C}^N) \mid K \subseteq H\}$$

and let π_1 and π_2 denote the projections onto the first and second factors, respectively. Then, the dimension of the fibres of π_1 is always 1, and thus $\dim W \leqslant N-2$. Therefore, the dimension of the closure of the image of π_2 is $\leqslant N-2$, and we are finished. □

We must now assume that the reader is familar with the basic results on the derived category, perverse sheaves, and on the sheaves of neighbouring cycles and vanishing cycles as described in [2, 5, 7, 10, 16, 22]. We shall also use some results from stratified Morse theory with coefficients in a complex of sheaves; these results are summarized in [12, II.6.A].

Definition 1.15. We let $\mathbf{F}^\bullet$ denote a bounded, constructible complex of sheaves of $\mathbb{Z}$-modules. If $\mathbf{F}^\bullet$ is a perverse sheaf on the s-dimensional space X, then its non-zero cohomology groups lie in non-positive dimensions. For many of our purposes, it will be more convenient to shift the complex so that the cohomology is in non-negative dimensions. Hence, we define $\mathbf{F}^\bullet$ on X to be *positively perverse* if and only if $\mathbf{F}^\bullet[s]$ is perverse on X.

For instance, if X is a connected local complete intersection, then the constant sheaf $\mathbb{Z}^\bullet_X$ is positively perverse on X (see [20]).

Note that if X is not pure-dimensional, then the notion of positively perverse does not localize well.

If $f : X \to \mathbb{C}$ is an analytic map and $\mathbf{F}^\bullet$ is a bounded, constructible complex on X, then we denote the sheaves of neighbouring and vanishing cycles of $\mathbf{F}^\bullet$ along f by $\psi_f\mathbf{F}^\bullet$ and $\phi_f\mathbf{F}^\bullet$, respectively.

We will need the following basic, well-known results.

PROPOSITION 1.16. (See [2, 5, 16, 23].) *Let $\mathcal{S}$ be a Whitney stratification of an s-dimensional complex space X, and let $\mathbf{F}^\bullet$ be a bounded complex on X which is constructible with respect to $\mathcal{S}$. Suppose we have $f : (X, \mathbf{0}) \to (\mathbb{C}, 0)$. Then, the cohomology sheaves $\mathbf{H}^*(\phi_f\mathbf{F}^\bullet)$ are supported only on $\Sigma_{\mathcal{S}} f$.*

Moreover, if $\mathbf{F}^\bullet$ *is positively perverse and* $\dim X \cap V(f) = s - 1$, *then* $\phi_f \mathbf{F}^\bullet$ *is positively perverse on* $X \cap V(f)$ (*with no shifts*) *and* $\mathbf{H}^i(\phi_f \mathbf{F}^\bullet)$ *is supported only for those i such that* $s - 1 - \dim \Sigma_{\mathscr{S}} f \leqslant i \leqslant s - 1$.

In particular, if $\dim_0 \Sigma_{\mathscr{S}} f = 0$ *and* $\mathbf{F}^\bullet$ *is positively perverse, then* $\mathbf{H}^i(\phi_f \mathbf{F}^\bullet)_0$ *is* (*possibly*) *non-zero only in dimension* $s - 1$.

Proposition 1.17. (See [12, II.6.A].) *Let* $\mathscr{S} = \{S_\alpha\}$ *be a Whitney stratification* (*with connected strata*) *of the s-dimensional space X, let* $\mathbf{F}^\bullet$ *be a bounded complex of* $\mathbb{Z}$*-modules which is constructible with respect to* $\mathscr{S}$, *and let* $\mathbf{p} \in S_\alpha \in \mathscr{S}$.

If $f: X \to \mathbb{C}$ *is any analytic function which has* $\mathbf{p}$ *as a non-degenerate critical point, then the hypercohomology of the local Morse data of f at* $\mathbf{p}$ *is isomorphic to the shifted hypercohomology of the normal Morse data to* S_α, *i.e. for all i,*

$$\mathbb{H}^i(\mathring{B}_\varepsilon(\mathbf{p}), \mathring{B}_\varepsilon(\mathbf{p}) \cap f^{-1}(v); \mathbf{F}^\bullet) \cong \mathbb{H}^{i-d_\alpha}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet),$$

where $d_\alpha = \dim S_\alpha$ *and* $0 < |v - f(\mathbf{p})| \ll \varepsilon \ll 1$.

Definition 1.18. Let $\mathscr{S} = \{S_\alpha\}$ be a Whitney stratification (with connected strata) of the s-dimensional space X, let $\mathbf{F}^\bullet$ be a bounded complex of $\mathbb{Z}$-modulus which is constructible with respect to $\mathscr{S}$, and let $\mathbf{p} \in S_\alpha \in \mathscr{S}$.

The *characteristic cycle, $Ch(\mathbf{F}^\bullet)$, of $\mathbf{F}^\bullet$* in the conormal bundle over $\mathscr{U}$, $T^*\mathscr{U}$, is the linear combination $\sum_\alpha m_\alpha(\mathbf{F}^\bullet)\overline{T^*_{S_\alpha}\mathscr{U}}$, where $\overline{T^*_{S_\alpha}\mathscr{U}}$ denotes the closure of the conormal bundle to S_α in $T^*\mathscr{U}$ and the $m_\alpha(\mathbf{F}^\bullet)$ are integers given by

$$\begin{aligned} m_\alpha(\mathbf{F}^\bullet) &:= (-1)^{s-1}\chi(\phi_{f_{|X}}\mathbf{F}^\bullet)_\mathbf{p} = (-1)^{s-d_\alpha-1}\chi(\phi_{f_{|N}}\mathbf{F}^\bullet_{|N})_\mathbf{p} \\ &= (-1)^{s-d_\alpha}\chi(\mathbb{H}^*(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet)) \end{aligned}$$

for any point $\mathbf{p}$ in the d_α-dimensional stratum S_α, with normal slice N to S_α at $\mathbf{p}$, and any function $f: (\mathscr{U}, \mathbf{p}) \to (\mathbb{C}, 0)$ which has $\mathbf{p}$ as a non-degenerate critical point. This cycle is independent of all the choices made. We refer to $m_\alpha(\mathbf{F}^\bullet)$ as the *normal index* of S_α relative to $\mathbf{F}^\bullet$, and write simply m_α if the complex $\mathbf{F}^\bullet$ is clearly specified.

Remark 1.19. If $\mathbf{F}^\bullet$ is positively perverse on the s-dimensional space X, then, by Proposition 1.16,

$$H^{i-1}(\phi_{f_{|N}}\mathbf{F}^\bullet_{|N})_\mathbf{p} = \mathbb{H}^i(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet) = 0$$

unless $i = s - d_\alpha$. Thus, if $\mathbf{F}^\bullet$ is positively perverse, all of the normal indices are non-negative. This is true whether X is pure-dimensional or not. Note, however, that the constant sheaf on a non-pure-dimensional space will not be positively perverse.

There is the following important result:

Theorem 1.20 (A'Campo [1]). *Let* $\mathbf{A}$ *be a bounded, constructible complex of* $\mathbb{C}$*-vector spaces on X and* $f: X \to \mathbb{C}$. *The monodromy automorphism* $\psi_f \mathbf{A}^\bullet \to \psi_f \mathbf{A}^\bullet$ *induces a map on cohomology sheaves*

$$\mathbf{H}^*(\psi_f \mathbf{A}^\bullet)_\mathbf{p} \xrightarrow{\mu} \mathbf{H}^*(\psi_f \mathbf{A}^\bullet)_\mathbf{p}.$$

If $\mathfrak{m}_{\mathbf{p}}$ *denotes the maximal ideal of* X *at* $\mathbf{p}$ *and* $f \in \mathfrak{m}_{\mathbf{p}}^2$, *then the Lefschetz number of* $\mathbf{H}^*(\psi_f \mathbf{A}^\bullet)_{\mathbf{p}} \xrightarrow{\mu} \mathbf{H}^*(\psi_f \mathbf{A}^\bullet)_{\mathbf{p}}$ *equals* 0, *i.e.*

$$\sum_i (-1)^i \, Trace\{\mathbf{H}^i(\psi_f \mathbf{A}^\bullet)_{\mathbf{p}} \xrightarrow{\mu} \mathbf{H}^i(\psi_f \mathbf{A}^\bullet)_{\mathbf{p}}\} = 0.$$

2. THE RELATIVE HYPERCOHOMOLOGY SPLITTING LEMMA

In this section, we prove a simple, but powerful, lemma which says that certain relative hypercohomology modules split as direct sums. The proof involves only the stratified Morse theory of Goresky and MacPherson [12] together with an easy homotopy argument. This homotopy argument is a general form of that used by Siersma in [38].

Our conclusion is a refinement of that arrived at via stratified Morse theory, for it tells us that there is no cancellation going on between the various local Morse data.

The principal example of a space on which we want to apply our lemma is a complex analytic space intersected with a closed ball in some affine space; hence, we will have both real and complex Whitney strata. Therefore, we make the following definition.

Definition 2.1. Let X be an analytic space and let Y be a subset of X. Then, Y is a *complex Whitney stratified set with boundary* if there exists a real Whitney stratification $\mathscr{R}$ of Y together with a subcollection of strata $\mathscr{S} \subseteq \mathscr{R}$ such that Y equals the closure in Y of $\bigcup_{S \in \mathscr{S}} S$, $\mathscr{S}$ is a complex Whitney stratification of $\bigcup_{S \in \mathscr{S}} S$ (in particular, all of the strata of $\mathscr{S}$ are complex analytic) and, for all $S \in \mathscr{S}$ and $R \in \mathscr{R} - \mathscr{S}$, $\bar{R} \cap S = \emptyset$.

We refer to the strata in $\mathscr{S}$ as the *interior strata* of the complex Whitney stratified set with boundary.

Below, we use our notation from Definition 1.5 for the normal Morse data to complex strata. Note that the normal Morse data is well-defined for the interior strata of a complex Whitney stratified set with boundary.

LEMMA 2.2. *Let* X *be a complex analytic space and let* $f: X \to \mathbb{C}$ *be a complex analytic function. Let* Y *be a complex Whitney stratified subset of* X *with real Whitney stratification* $\mathscr{R}$ *and interior strata* $\mathscr{S}$. *Let* $\mathbf{F}^\bullet$ *be a bounded complex of sheaves of* $\mathbb{Z}$*-modules which is constructible with respect to* $\mathscr{R}$.

Suppose that there exists an open complex disc $\mathring{\mathbb{D}}_\eta \subseteq \mathbb{C}$ *such that* $f|_Y$ *is proper over* $\mathring{\mathbb{D}}_\eta$. *Let* $v \in \mathring{\mathbb{D}}_\eta$.

(A) *Suppose that, with respect to* $\mathscr{R}$, *the function* $f|_Y$ *has a finite collection of stratified critical values*, $\{v_0, v_1, \ldots, v_m\}$, *in* $\mathring{\mathbb{D}}_\eta$. *Let* $\mathring{\mathbb{D}}_0, \mathring{\mathbb{D}}_1, \ldots, \mathring{\mathbb{D}}_m$ *be a collection of disjoint discs in* $\mathring{\mathbb{D}}_\eta$, *centred at* $v_0, v_1, \ldots, v_m$, *respectively. For all i between* 0 *and m, let* ξ_i *be in* $\mathring{\mathbb{D}}_i - \{v_i\}$.

Then, the relative hypercohomology modules of the pair $(Y \cap f^{-1}(\mathring{\mathbb{D}}_\eta), Y \cap f^{-1}(v))$ *with coefficients in* $\mathbf{F}^\bullet$ *split as direct sums of contributions over the various critical values; specifically, for all j*,

$$\mathbb{H}^j(Y \cap f^{-1}(\mathring{\mathbb{D}}_\eta), Y \cap f^{-1}(v); \mathbf{F}^\bullet) \cong \bigoplus_{v_i \neq v} \mathbb{H}^j(Y \cap f^{-1}(\mathring{\mathbb{D}}_i), Y \cap f^{-1}(\xi_i); \mathbf{F}^\bullet).$$

(B) *Suppose that for all* $R \in \mathscr{R} - \mathscr{S}$, $f|_R$ *has no critical points over* $\mathring{\mathbb{D}}_\eta - \{v\}$ *and, if* $R \in \mathscr{S}$, *then the stratified critical points of f which occur on* $R \cap f^{-1}(\mathring{\mathbb{D}}_\eta - \{v\})$ *are all non-degenerate.*

Then, the relative hypercohomology modules of the pair $(Y \cap f^{-1}(\mathring{\mathbb{D}}_\eta), Y \cap f^{-1}(v))$ *with coefficients in* $\mathbf{F}^\bullet$ *split as direct sums of powers of the shifted hypercohomology of the normal Morse data to the complex strata; specifically, for all j,*

$$\mathbb{H}^j(Y \cap f^{-1}(\mathring{\mathbb{D}}_\eta), Y \cap f^{-1}(v); \mathbf{F}^\bullet) \cong \bigoplus_{S_\alpha \in \mathscr{S}} (\mathbb{H}^{j-d_\alpha}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))^{k_\alpha(v)}$$

where $d_\alpha = \dim S_\alpha$ *and* $k_\alpha(v) =$ *the number of critical points,* $\mathbf{p}$, *of* $f|_{S_\alpha}$ *over* $\mathring{\mathbb{D}}_\eta - \{v\}$.

Proof. (A) Despite the complex of coefficients, this argument is essentially that of [38]. The function f is a proper, stratified submersion over $\mathring{\mathbb{D}}_\eta$ except at the isolated critical values. Hence, any smooth homotopy in the base which leaves the critical values fixed lifts to a stratum-preserving homotopy in Y.

Let A denote a subset of $\mathring{\mathbb{D}}_\eta$ which consists of a collection of disjoint smaller discs, one around each critical point, which are all connected to v by non-intersecting paths. Now, smoothly deform $\mathring{\mathbb{D}}_\eta$ to a sufficiently small, smooth tubular neighbourhood of A. By the previous paragraph, this homotopy lifts to a stratum-preserving homotopy and, hence, yields isomorphisms on hypercohomology. By utilizing 8.4.7 of [16], we see that we may continue to deform the base all the way down to A and still obtain induced isomorphisms on the hypercohomology of Y.

Let U be a neighbourhood of v in A which is so small that it misses each of the small discs around the critical values. Then,

$$\begin{aligned}\mathbb{H}^j(Y \cap f^{-1}(\mathring{\mathbb{D}}_\eta), Y \cap f^{-1}(v); \mathbf{F}^\bullet) &\cong \mathbb{H}^j(Y \cap f^{-1}(A), Y \cap f^{-1}(U); \mathbf{F}^\bullet)\\ &\cong \mathbb{H}^j(Y \cap f^{-1}(A - \{v\}), Y \cap f^{-1}(U - \{v\}); \mathbf{F}^\bullet)\\ &\cong \bigoplus_{v_i \neq v} \mathbb{H}^j(Y \cap f^{-1}(\mathring{\mathbb{D}}_i), Y \cap f^{-1}(\xi_i); \mathbf{F}^\bullet)\end{aligned}$$

where the first isomorphism is via homotopy (and, if v is a critical value, by 8.4.7 of [16]), the second isomorphism follows from excision, and the third isomorphism follows from deforming the pair $(A - \{v\}, U - \{v\})$, which is a disjoint collection of discs with line segments emanating from them modulo the ends of the line segments.

(B) We may assume that the critical points of f have distinct critical values, for—if not—we may perturb f slightly; as non-degenerate critical points are stable, this affects nothing.

Considering that we already have part (A), all that remains to be shown is that if $S_\alpha \in \mathscr{S}$ and $\mathbf{p} \in S_\alpha \cap f^{-1}(\mathring{\mathbb{D}}_\eta - \{v\})$ is a non-degenerate stratified critical point, then

$$\mathbb{H}^j(Y \cap f^{-1}(\mathring{\mathbb{D}}), Y \cap f^{-1}(f(\mathbf{p})); \mathbf{F}^\bullet) \cong \mathbb{H}^{j-d_\alpha}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet)$$

where $\mathring{\mathbb{D}}$ is a small disc centred around $f(\mathbf{p})$ and $d_\alpha = \dim S_\alpha$. But this follows immediately from Proposition 1.17. $\square$

3. STRATIFIED ISOLATED SINGULARITIES

The primary result of this section is to analyze the structure of the stalk cohomology of the sheaf of vanishing cycles of a stratified isolated singularity; this is an application of the relative hypercohomology splitting lemma and, as such, the conclusion is that the stalk cohomology of the sheaf of vanishing cycles is a direct sum of powers of contributions from the normal data to the strata. In the theorem, we describe the exponents which appear in this direct sum; however, we shall defer the proof of the exponent statement until Section 4. In the constant coefficient case and without the exponent statement, the result appears in [23, 38, 40]; however, each of these papers deals with the more difficult description of the homotopy-type of the Milnor fibre. On the other hand, since these three aforementioned papers do not provide any means of calculating the exponents in the direct sum, they are not terribly useful for performing calculations.

We shall need the following lemma.

LEMMA 3.1. *Let $\mathscr{S}$ be a Whitney stratification of a complex space X, let $S_\alpha \in \mathscr{S}$, and let k be an integer. Then, there exists a bounded complex of $\mathbb{Z}$-modules, $\mathbf{U}^\bullet(X, S_\alpha; k)$, which is constructible with respect to $\mathscr{S}$ such that the normal indices, $m_\beta(\mathbf{U}^\bullet(X, S_\alpha; k))$, are all 0 except when $\beta = \alpha$, and $m_\alpha(\mathbf{U}^\bullet(X, S_\alpha; k)) = k$.*

Proof. First, note that it suffices to construct $\mathbf{U}^\bullet(X, S_\alpha; 1)$ for each S_α; for then, we can let $\mathbf{U}^\bullet(X, S_\alpha; k) = (\mathbf{U}^\bullet(X, S_\alpha; 1))^k$ if k is non-negative, or $\mathbf{U}^\bullet(X, S_\alpha; k) = (\mathbf{U}^\bullet(X, S_\alpha; 1)\,[1])^{-k}$ if k is negative.

We proceed with the proof by induction on the dimension of X.

If $\dim X = 0$, then the stratum S_α is merely a point. We let $\mathbf{U}^\bullet(X, S_\alpha; 1)$ be the extension by zero of the constant sheaf on S_α; that is, if $j: S_\alpha \hookrightarrow X$ denotes the inclusion, then we let $\mathbf{U}^\bullet(X, S_\alpha; 1) = j_!\mathbb{Z}^\bullet_{S_\alpha}$.

Now, suppose that $\dim X = s$. Let $j: \overline{S_\alpha} \hookrightarrow X$ denote the inclusion, and let $s' := \dim S_\alpha$.

If $s' < s$, then by our inductive hypothesis, for any integer k, we have already constructed $\mathbf{U}^\bullet(\bar{S}_\alpha, S_\alpha; k)$ on $\bar{S}_\alpha$. Then, we may define $\mathbf{U}^\bullet(X, S_\alpha; k) := j_!\mathbf{U}^\bullet(\bar{S}_\alpha, S_\alpha; k)\,[s - s']$.

Suppose though that $s = s'$. Consider the complex $\mathbf{F}^\bullet := j_!\mathbb{Z}^\bullet_{S_\alpha}$; this complex is constructible with respect to $\mathscr{S}$, has $m_\alpha = 1$, and has $m_\beta = 0$ for any S_β not contained in the closure of S_α. We shall now "correct" $\mathbf{F}^\bullet$ on the strata contained in $\bar{S}_\alpha$.

If $S_\beta \subseteq \bar{S}_\alpha$ and $S_\beta \neq S_\alpha$, then $\dim S_\beta < s$ and, by our previous paragraph, we have already constructed the complex $\mathbf{U}^\bullet(X, S_\beta; k)$. So, define

$$\mathbf{U}^\bullet(X, S_\alpha; 1) := \mathbf{F}^\bullet \oplus \bigoplus_{\substack{S_\beta \subseteq \bar{S}_\alpha \\ S_\beta \neq S_\alpha}} \mathbf{U}^\bullet(X, S_\beta;\, -m_\beta(\mathbf{F}^\bullet)). \qquad \square$$

We will now prove the main theorem of this section. For notational convenience, we assume throughout the rest of this section that X is an analytic subset of some $\mathbb{C}^N$ and that $\mathbf{0} \in X$.

THEOREM 3.2. *Let $\mathscr{S} = \{S_\alpha\}$ be a Whitney stratification of X, and suppose that $f: (X, \mathbf{0}) \to (\mathbb{C}, 0)$ has a stratified isolated critical point at $\mathbf{0}$.*

Then, there exists a unique set, $\{k_\alpha\}$, of non-negative integers such that, for all bounded complexes, $\mathbf{F}^\bullet$, of $\mathbb{Z}$-modules on X which are constructible with respect to $\mathscr{S}$, and, for all i,

$$H^{i-1}(\phi_f\mathbf{F}^\bullet)_\mathbf{0} \cong \mathbb{H}^i(B_\varepsilon \cap X, F_{f,\mathbf{p}}; \mathbf{F}^\bullet) \cong \mathbb{H}^i(B_\varepsilon \cap X \cap f^{-1}(\mathring{\mathbb{D}}_\eta), B_\varepsilon \cap X \cap f^{-1}(v); \mathbf{F}^\bullet)$$

$$\cong \bigoplus_\alpha (\mathbb{H}^{i-d_\alpha}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))^{k_\alpha}$$

where

$$d_\alpha = \dim S_\alpha,$$

B_ε is a sufficiently small closed ball of radius ε centred at the origin in the ambient affine space; $\mathring{\mathbb{D}}_\eta$ is an open disc of radius η, $0 < \eta \ll \varepsilon$, centred at the origin;

$$v \in \mathring{\mathbb{D}}_\eta^*,$$

$F_{f,\mathbf{0}}$ is the Milnor fibre of f at $\mathbf{0}$ or, more precisely, $F_{f,\mathbf{0}} = B_\varepsilon \cap X \cap f^{-1}(v)$.

Moreover, $k_\alpha = 1$ if $S_\alpha = \{\mathbf{0}\}$ and, if $S_\alpha \neq \{\mathbf{0}\}$, then for a generic choice of linear forms, L, on the ambient affine space, $\tilde{\Gamma}_{f,L}(\mathscr{S})$ is one-dimensional (or empty) at the origin and

$$k_\alpha = (\tilde{\Gamma}^1_{f,L}(S_\alpha) \cdot V(f))_\mathbf{0} - (\tilde{\Gamma}^1_{f,L}(S_\alpha) \cdot V(L))_\mathbf{0}.$$

The integer k_α is precisely the number of non-degenerate critical points of a small perturbation of f by L which occur near the origin on the stratum S_α; more precisely, for all sufficiently small $\varepsilon > 0$, for all complex t such that $0 < |t| \ll \varepsilon$, k_α equals the number of critical points of $f + tL$ in $\mathring{B}_\varepsilon \cap S_\alpha$.

Proof. The first isomorphism is by definition; the second isomorphism is essentially the argument of Milnor [37, 5.11], but in the stratified case follows from [12, II.2.4]. It is the final isomorphism which requires a proof.

We first prove the existence of the k_α.

By [23, 2.2], for a generic choice of linear form, L, for all sufficiently small $\varepsilon > 0$, for all complex t such that $0 < |t| \ll \varepsilon$, $f + tL$ has only non-degenerate interior critical points in $B_\varepsilon \cap X$.

Now, pick such an L and such an ε, and also pick ε small enough to define the Milnor fibre in $B_\varepsilon \cap X$. Fix a complex $v \neq 0$ small enough so that $B_\varepsilon \cap X \cap f^{-1}(v)$ is the Milnor fibre of f at the origin.

As $f^{-1}(v)$ transversely intersects the strata of the compact set $B_\varepsilon \cap X$, the openness of transversality implies that, for all small complex t,

$$\mathbb{H}^i(B_\varepsilon \cap X, B_\varepsilon \cap X \cap f^{-1}(v); \mathbf{F}^\bullet) \cong \mathbb{H}^i(B_\varepsilon \cap X, B_\varepsilon \cap X \cap (f + tL)^{-1}(v); \mathbf{F}^\bullet).$$

Now, the existence of the k_α follows from Lemma 2.2(B).

We now prove the uniqueness of the k_α. Let $s := \dim X$. By taking Euler characteristics, we find that, for all $\mathbf{F}^\bullet$,

$$\chi(\phi_f \mathbf{F}^\bullet)_\mathbf{0} = \sum_\alpha (-1)^{d_\alpha - 1} k_\alpha \cdot \chi(\mathbb{H}^*(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet)) = \sum_\alpha (-1)^{s-1} k_\alpha m_\alpha(\mathbf{F}^\bullet).$$

Thus, fixing a stratum, S_γ and applying our lemma, we find that

$$\chi(\phi_f \mathbf{U}^\bullet(X, S_\gamma; 1))_\mathbf{0} = \sum_\alpha (-1)^{s-1} k_\alpha m_\alpha(\mathbf{U}^\bullet(X, S_\gamma; 1)) = (-1)^{s-1} k_\gamma.$$

Therefore, $k_\gamma = (-1)^{s-1} \chi(\phi_f \mathbf{U}^\bullet(X, S_\gamma; 1))_\mathbf{0}$.

Clearly, if $S_\alpha = \{\mathbf{0}\}$, then $k_\alpha = 1$; we give the proof of the intersection number characterization of the other k_α in Corollary 4.5. □

COROLLARY 3.3. *Let $\mathscr{S} = \{S_\alpha\}$ be a Whitney stratification of X, and suppose that $f:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ has a stratified isolated critical point at $\mathbf{0}$. Let $\mathbf{F}^\bullet$ be a positively perverse complex of $\mathbb{Z}$-modules on X which is constructible with respect to $\mathscr{S}$.*

Then, $H^{i-1}(\phi_f \mathbf{F}^\bullet)_\mathbf{0}$ is zero for all i except, perhaps, for when $i = s$. When $i = s$, we have the formula

$$\operatorname{rank} H^{s-1}(\phi_f \mathbf{F}^\bullet)_\mathbf{0} = m_\mathbf{0}(\mathbf{F}^\bullet) + \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)((\tilde{\Gamma}^1_{f,L}(S_\alpha) \cdot V(f))_\mathbf{0} - (\tilde{\Gamma}^1_{f,L}(S_\alpha) \cdot V(L))_\mathbf{0})$$

where L is a generic linear form, m_α denotes the normal index of S_α (recall Definition 1.18) and m_0 denotes the normal index of the origin (considered as a point stratum).

Moreover, all of the normal indices and each of the quantities

$$((\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0)$$

are non-negative.

Proof. This follows immediately from the theorem, Definition 1.18, and Remark 1.19. □

Remark 3.4. The statement of Theorem 3.2 deals with all of the strata on more or less equal footing. However, the theorem has a nicer feel to it if one splits off the information of what happens at the origin, as we did in Corollary 3.3.

We may certainly assume that the origin is a stratum. Then, the conclusion of Theorem 23.2 can be rewritten as

$$H^{i-1}(\phi_f\mathbf{F}^\bullet)_0 \cong \mathbb{H}^i(B_\varepsilon\cap X, F_{f,0}; \mathbf{F}^\bullet) \cong \mathbb{H}^i(B_\varepsilon\cap X, \mathbb{L}_{X,0}; \mathbf{F}^\bullet)\oplus \bigoplus_{S_\alpha\neq\{0\}} (\mathbb{H}^{i-d_\alpha}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))^{k_\alpha}$$

where $\mathbb{L}_{X,0}$ is the link of X at the origin (defined inside B_ε of course). In other words, the relative hypercohomology of the Milnor fibre of a function with a stratified isolated singularity always has the relative hypercohomology of the complex link as a direct summand. Of course, if we do not need to select the origin as a Whitney stratum, then $\mathbb{H}^i(B_\varepsilon\cap X, \mathbb{L}_{X,0}; \mathbf{F}^\bullet) = 0$ for all i.

It is possible to be precise about how generic the linear form, L, must be in Theorem 3.2 and Corollary 3.3. We need for d_0L to be non-degenerate with respect to $\mathscr{S}$, so that L actually determines the complex link at the origin. Letting $\mathscr{T}$ denote the good stratification of X induced by f (recall Definition 1.2), we also need for L to be prepolar with respect to $\mathscr{T}$ relative to f so that we may use neighbourhoods of a certain form to describe the Milnor fibre (see Lemma 4.1) and so that $L: X\to\mathbb{C}$ has a stratified isolated singularity. Finally, to obtain the equality

$$k_\alpha = (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0$$

we need for L to be decent with respect to $\mathscr{S}$ relative to f, and we need f to be decent with respect to $\mathscr{S}$ relative to L (see Corollary 4.5).

One final note before we give some examples: we have not explicitly said anything in the theorem about only summing over those strata which contain the origin in their closure—though certainly these are the only strata which can contribute to the vanishing cycles at the origin. This fact is already encoded in the statement that $k_\alpha = (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0$, for both of these intersection numbers will be 0 if the origin is not in the closure of S_α.

Example 3.5. Consider the classic situation studied by Milnor in [37]—the case of a polynomial $f:(\mathbb{C}^{n+1}, \mathbf{0})\to(\mathbb{C}, 0)$ with an isolated singularity at the origin. We consider $\mathbb{C}^{n+1}$ as being stratified with a single stratum, and we use the constant $\mathbb{Z}$ sheaf of coefficients.

Then, we have a single stratum $S_\alpha = \mathbb{C}^{n+1}$, $\mathbb{L}_\alpha = \emptyset$, $d_\alpha = n+1$, and $\tilde{\Gamma}^1_{f,L}(S_\alpha)$ is the ordinary relative polar curve of Lê and Teissier, which we denote simply by $\Gamma^1_{f,L}$.

In this setting, the conclusion of Theorem 3.2 is that

$$H^{i-1}(\phi_f \mathbb{Z}^\bullet_{\mathbb{C}^{n+1}})_0 \cong H^i(B_\varepsilon, F_{f,0}; \mathbb{Z}) \cong (H^{i-(n+1)}(point; \mathbb{Z}))^\mu,$$

where $\mu = (\Gamma^1_{f,L} \cdot V(f))_0 - (\Gamma^1_{f,L} \cdot V(L))_0$ for generic linear L.

We have not written μ here by coincidence; it is an easy exercise (see [18] or [33, 5.8]) to show in this case that $(\Gamma^1_{f,L} \cdot V(f))_0 - (\Gamma^1_{f,L} \cdot V(L))_0 = (\Gamma^1_{f,L} \cdot V(\partial f/\partial L))_0$, which is the Milnor number of f at the origin. Thus, we find that $H^i(B_\varepsilon, F_{f,0}; \mathbb{Z})$ is zero, except in degree $n+1$, where it is free-Abelian of rank μ—this is Milnor's result on the level of cohomology.

Example 3.6. The next easiest example is where we once again consider the constant sheaf but, this time, on a space, X, with an isolated singularity at the origin. In this case, in a neighbourhood of the origin, we may stratify X by taking as strata the origin together with the irreducible components of X minus the origin.

If $f:(X, \mathbf{0}) \to (\mathbb{C}, 0)$ has an isolated critical point at the origin, then we conclude from Theorem 3.2 that

$$H^{i-1}(\phi_f \mathbb{Z}^\bullet_{\mathbb{C}^{n+1}})_0 \cong H^i(B_\varepsilon \cap X, F_{f,0}; \mathbb{Z}) \cong H^i(B_\varepsilon \cap X, \mathbb{L}_{X,0}; \mathbb{Z}) \oplus \bigoplus_{C \neq \{0\}} (H^{i-\dim C}(point; \mathbb{Z}))^{k_C}$$

where the sum is over the irreducible components, C, of X at $\mathbf{0}$ and

$$k_C = (\tilde{\Gamma}^1_{f,L}(C) \cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(C) \cdot V(L))_0$$

for generic linear L.

As was discussed in Remark 3.4, the amount of genericity that we require of L is that $d_0 L$ is non-degenerate, that L is prepolar with respect to the good stratification induced by f, and that f and L are decent with respect to each other. However, the decency conditions are automatic here since the origin is the only stratum which is contained in the closure of another stratum. Also, prepolarity is easy in this case too; L is prepolar if and only if $L: X \to \mathbb{C}$ has an isolated critical point at the origin and the critical locus of the map $(f, L)|_{X-\mathbf{0}}$ has no points contained in $V(L)$ near the origin.

Let us look at one of the examples from Section 1. Example 1.8 is not a stratified isolated singularity, but Example 1.9 is. So, consider the example where $X := V(x^2 + y^2 + z^2) \subseteq \mathbb{C}^3$ and $f := x^3 + yz$.

We use y for our linear form in the computation of the exponents. One easily checks that $d_0 y$ is non-degenerate, and that y is prepolar with respect to the good stratification induced by f.

Therefore, we conclude from Theorem 3.2 and Example 1.9 that

$$H^{i-1}(\phi_f \mathbb{Z}^\bullet_X)_0 \cong H^i(B_\varepsilon \cap X, F_{f,0}; \mathbb{Z}) \cong H^i(B_\varepsilon \cap X, F_{y,0}; \mathbb{Z}) \oplus (H^{i-2}(point; \mathbb{Z}))^k$$

where k equals

$$(V(x, y^2 + z^2) + V(3xz - y, x^2 + y^2 + z^2) \cdot V(x^3 + yz))_0 - (V(x, y^2 + z^2)$$
$$+ V(3xz - y, x^2 + y^2 + z^2) \cdot V(y))_0.$$

Now, one computes with relative ease that:

$$(V(x, y^2 + z^2) \cdot V(x^3 + yz))_0 = 4$$

$$(V(3xz - y, x^2 + y^2 + z^2) \cdot V(x^3 + yz))_0 = 6$$

$$(V(x, y^2 + z^2)\cdot V(y))_0 = 2$$

$$(V(3xz - y, x^2 + y^2 + z^2)\cdot V(y))_0 = 4.$$

Hence, $k = (4 + 6) - (2 + 4) = 4$.

Moreover, the Milnor fibre $F_{y,0}$ is homeomorphic to the Milnor fibre of $x^2 + z^2$ inside $\mathbb{C}^2$, and so is homotopy-equivalent to a circle. Thus, $H^i(B_\varepsilon \cap X, F_{y,0}; \mathbb{Z}) = 0$ unless $i = 2$ and

$$H^2(B_\varepsilon \cap X, F_{y,0}; \mathbb{Z}) \cong \mathbb{Z}.$$

Therefore, we find that $H^{i-1}(\phi_f \mathbb{Z}_X^\bullet)_0 = H^i(B_\varepsilon \cap X, F_{f,0}; \mathbb{Z}) = 0$ unless $i = 2$ and

$$H^1(\phi_f \mathbb{Z}_X^\bullet)_0 \cong H^2(B_\varepsilon \cap X, F_{f,0}; \mathbb{Z}) \cong \mathbb{Z}^5.$$

It is not a fluke that the vanishing cohomology is concentrated in degree 1—this follows from Corollary 3.3 since the constant sheaf on a local complete intersection is positively perverse [20].

Example 3.7. Let us look at an isolated singularity on which the constant sheaf is not positively perverse. The easiest such example is two coordinate 2-planes in $\mathbb{C}^4$ which intersect only at the origin.

Using (w, x, y, z) as coordinates on $\mathbb{C}^4$, let $X = V(w, x) \cup V(y, z)$. Then, at the origin, X fails to be a local complete intersection and, as we shall see, $\mathbb{Z}_X^\bullet$ is not positively perverse. Naturally, we stratify X by taking as strata $\{\mathbf{0}\}$, $C := V(w, x) - \{\mathbf{0}\}$ and $D := V(y, z) - \{\mathbf{0}\}$.

Let $f: X \to \mathbb{C}$ be given by $f = w^2 + x^2 + y^2 + z^2$. Then, f has a stratified isolated singularity at the origin. Moreover, the Milnor fibre of f is easy to describe: $F_{f,0}$ has two connected components, each of which is homotopy-equivalent to a 1-sphere. It follows that the vanishing cohomology is given by $H^0(\phi_f \mathbb{Z}_X^\bullet)_0 \cong \mathbb{Z}$, $H^1(\phi_f \mathbb{Z}_X^\bullet)_0 \cong \mathbb{Z}^2$, and is zero in all other degrees.

Note that this shows that $\mathbb{Z}_X^\bullet$ is not positively perverse, for the vanishing cycle cohomology is not concentrated in degree 1.

We wish to see that this agrees with the formula of Theorem 3.2.

The linear form $L := w + x + y + z$ certainly satisfies the criteria given in Remark 3.4.

The complex link of $\mathbf{0}$ is clearly just two disjoint complex discs. Thus, the normal Morse data to the origin has hypercohomology given by

$$\mathbb{H}^1(c(\mathbb{L}_{\{0\}}), \mathbb{L}_{\{0\}}; \mathbb{Z}_X^\bullet) \cong H^1(c(\mathbb{L}_{\{0\}}), \mathbb{L}_{\{0\}}; \mathbb{Z}) \cong \mathbb{Z}$$

and is zero in all other degrees.

The normal data to both C and D is just $(\mathit{point}, \emptyset)$. Moreover, one easily calculates that $\Gamma^1_{f,L}(C) = V(w, x, y - z)$ and $\Gamma^1_{f,L}(D) = V(y, z, w - x)$, and

$$(\Gamma^1_{f,L}(C)\cdot V(f))_0 = 2 = (\Gamma^1_{f,L}(D)\cdot V(f))_0$$

and

$$(\Gamma^1_{f,L}(C)\cdot V(L))_0 = 1 = (\Gamma^1_{f,L}(D)\cdot V(L))_0.$$

Therefore, from Theorem 3.2, we find

$$H^0(\phi_f \mathbb{Z}_X^\bullet) \cong H^1(c(\mathbb{L}_{\{0\}}), \mathbb{L}_{\{0\}}; \mathbb{Z}) \cong \mathbb{Z}$$

and

$$H^1(\phi_f \mathbb{Z}_X^\bullet) \cong H^2(c(\mathbb{L}_{\{0\}}), \mathbb{L}_{\{0\}}; \mathbb{Z}) \oplus (H^0(\mathit{point}; \mathbb{Z}))^{(2-1)} \oplus (H^0(\mathit{point}; \mathbb{Z}))^{(2-1)} \cong 0 \oplus \mathbb{Z}^2 \cong \mathbb{Z}^2$$

and the vanishing cycles have zero cohomology in all other degrees. This agrees with our previous calculation.

Example 3.8. We wish to reconsider the above example with the same X, the same f, and the same L, but with a different complex of sheaves. Instead of the constant sheaf on X, we will use the intersection cohomology sheaf, $\mathbf{IC}_X^\bullet$, with constant coefficients (with topological indexing); this is always a positively perverse sheaf.

For our simple X, this intersection cohomology complex is easy to describe: it is the direct sum of the constant sheaf on each of the two components of X. Since this complex only differs from the constant sheaf at the origin, the cohomology of the Milnor fibre of f at $\mathbf{0}$ with intersection cohomology coefficients is exactly the same as the ordinary cohomology.

However, $H^*(\phi_f \mathbf{IC}_X^\bullet)_0$ takes the activity at the origin into account. Using the long exact hypercohomology sequence of the pair $(B_\varepsilon \cap X, F_{f,\mathbf{0}})$, one easily finds that the vanishing cohomology is given by $H^1(\phi_f \mathbf{IC}_X^\bullet)_0 \cong \mathbb{Z}^2$, and is zero in all other degrees.

Note that, since $\mathbf{IC}_X^\bullet$ is positively perverse, the vanishing cycle cohomology is concentrated in degree 1.

Now we want to see what effect the change of complexes has on our calculation from Example 3.7 and our application of Theorem 3.2.

Certainly, the exponents do not change, for they do not depend on the complex at all. Also, the hypercohomology of the normal data to the two 2-dimensional components does not change because we have not changed our complex away from the origin.

Thus, the only change in applying Theorem 3.2 is in calculating the hypercohomology of the normal data at the origin. Using the long exact sequence on hypercohomology of the pair $(B_\varepsilon, \mathbb{L}_{X,\mathbf{0}}) = (c(\mathbb{L}_{\{0\}}), \mathbb{L}_{\{0\}})$, we find that $\mathbb{H}^i(c(\mathbb{L}_{\{0\}}), \mathbb{L}_{\{0\}}; \mathbf{IC}_X^\bullet)$ is zero in all degrees.

Hence, we find that the result of Theorem 3.2 agrees with our direct calculation.

We will now prove a result which essentially says that if f has a stratified isolated singularity at the origin, then the complex link of $V(f)$ is the same as the Milnor fibre of f restricted to a generic hyperplane slice.

THEOREM 3.9. *Let $\mathscr{S}$ be a Whitney stratification of X and suppose that $f: X \to \mathbb{C}$ has a stratified isolated singularity at the origin. Let $g: X \to \mathbb{C}$ be prepolar with respect to the good stratification induced by f. Then, the Milnor fibre $F_{f|_{V(g)},\mathbf{0}}$ is stratified-homeomorphic to the Milnor fibre $F_{g|_{V(f)},0}$ with respect to the stratification induced on the Milnor fibres by the original stratification, $\mathscr{S}$.*

Proof. Let $\mathscr{S}'$ denote the stratification induced by f; that is, in a neighbourhood of the origin,

$$\mathscr{S}' = \{S_\alpha - V(f) \mid S_\alpha \in \mathscr{S}\} \cup \{S_\alpha \cap V(f) - \mathbf{0} \mid S_\alpha \in \mathscr{S}\} \cup \{\mathbf{0}\}.$$

Note that, as components contained in $V(f)$ are specifically removed from the polar variety $\Gamma_{f,g}(\mathscr{S}) = \Gamma_{f,g}(\mathscr{S}')$.

Let $\varepsilon > 0$ be so small that $B_\varepsilon \cap \Sigma_{\mathscr{S}} f \subseteq \{\mathbf{0}\}$ and small enough so that we may define $F_{f|_{V(g)},\mathbf{0}}$ and $F_{g|_{V(f)},\mathbf{0}}$ inside B_ε. As g is prepolar with respect to $\mathscr{S}'$, by Proposition 1.12, we may also pick ε so small that $B_\varepsilon \cap \Gamma_{f,g}(\mathscr{S}')$ equals $B_\varepsilon \cap \tilde{\Gamma}_{f,g}(\mathscr{S}')$ and is one-dimensional or empty. Finally, again because g is prepolar with respect to $\mathscr{S}'$, we may choose ε so small that, inside an open ball containing B_ε, $V(g)$ transversely intersects each of the strata in $\{S_\alpha \cap V(f) - \mathbf{0} \mid S_\alpha \in \mathscr{S}\}$ and so that the sphere ∂B_ε transversely intersects each of $S_\alpha \cap V(f) \cap V(g)$.

Fixing ε, what we need to show is that, for $0 < \eta,\ \nu \ll \varepsilon$, the map

$$B_\varepsilon \cap X - \psi^{-1}(\mathring{\mathbb{D}}_\eta \times \mathring{\mathbb{D}}_\nu - \psi(\tilde{\Gamma}^1_{f,g}(\mathscr{S})))$$
$$\downarrow \psi := (f, g)$$
$$\mathring{\mathbb{D}}_\eta \times \mathring{\mathbb{D}}_\nu - \psi(\tilde{\Gamma}^1_{f,g}(\mathscr{S}))$$

is a proper, stratified submersion with respect to $\mathscr{S}$.

It is easy to see that ψ has no interior stratified critical points; f and g each have at worst an isolated critical point at the origin, and we have explicitly removed the polar curve. Moreover, ψ has no critical points contained in $V(g)$ since $B_\varepsilon \cap \Gamma_{f,g}(\mathscr{S}) = B_\varepsilon \cap \tilde{\Gamma}_{f,g}(\mathscr{S})$. Finally, ψ has no critical points contained in $V(f)$ because g is prepolar with respect to $\mathscr{S}'$.

We must still show that ψ has no critical points on the boundary strata of $\partial B_\varepsilon \cap X$. Suppose to the contrary that, no matter how small we pick η and ν, ψ has stratified critical points on $\partial B_\varepsilon \cap X$. By the local finiteness of the stratification, we may assume that all of these critical points are contained in a single stratum S_α. Let $\tilde{f}$ and $\tilde{g}$ denote extensions of f and g, respectively, to the ambient affine space.

Then, we would have an infinite sequence $\mathbf{p}_i \in \partial B_\varepsilon \cap S_\alpha$ such that $f(\mathbf{p}_i) \to 0$, $g(\mathbf{p}_i) \to 0$, and

$$T_{\mathbf{p}_i} V(\tilde{f} - \tilde{f}(\mathbf{p}_i)) \cap T_{\mathbf{p}_i} V(\tilde{g} - \tilde{g}(\mathbf{p}_i)) \cap T_{\mathbf{p}_i} S_\alpha \subseteq T_{\mathbf{p}_i} \partial B_\varepsilon.$$

We may assume that the $\mathbf{p}_i$ converge to some $\mathbf{p} \in \partial B_\varepsilon$ in a stratum $S_\beta \subseteq \bar{S}_\alpha$. By Whitney's condition A, we may assume that $T_{\mathbf{p}_i} S_\alpha$ approaches some limit $\mathscr{T}$ which contains $T_{\mathbf{p}} S_\beta$. As $\mathbf{p}$ is not the origin, $V(\tilde{g})$, $V(\tilde{f})$, and $T_{\mathbf{p}} S_\beta$ intersect transversely at $\mathbf{p}$, and $T_{\mathbf{p}_i} V(\tilde{f} - \tilde{f}(\mathbf{p}_i)) \to T_{\mathbf{p}} V(\tilde{f})$ and $T_{\mathbf{p}_i} V(\tilde{g} - \tilde{g}(\mathbf{p}_i)) \to T_{\mathbf{p}} V(\tilde{g})$. But, all of this leads us to conclude that

$$V(\tilde{g}) \cap V(\tilde{f}) \cap T_{\mathbf{p}} S_\beta \subseteq V(\tilde{g}) \cap V(\tilde{f}) \cap \mathscr{T} \subseteq T_{\mathbf{p}} \partial B_\varepsilon.$$

This contradicts our assumption on the choice of ε, for we required that ∂B_ε transversely intersect $S_\beta \cap V(f) \cap V(g)$. □

Corollary 3.10. *Let $\mathscr{S}$ be a Whitney stratification of X, let $f: (X, \mathbf{0}) \to (\mathbb{C}, 0)$ have a stratified isolated singularity at the origin, and let $\mathfrak{m}_{X,\mathbf{0}}$ denote the maximal ideal of X at $\mathbf{0}$. If $f \in \mathfrak{m}^2_{X,\mathbf{0}}$, then the complex link of $V(f)$ at the origin does not have the complex cohomology of a point.*

Proof. This almost follows immediately from the result of A'Campo which we stated in Theorem 1.20. However, the generic linear form used to define the complex link of $V(f)$ need not be in the square of the maximal ideal of $V(f)$ at the origin; this means that we cannot apply A'Campo's result directly.

Let L be a linear form which is generic enough to give the complex link of $V(f)$ at the origin and generic enough to apply Theorem 3.9. Then, Theorem 3.9 tells us that the complex link of $V(f)$ at the origin is homeomorphic to the Milnor fibre at the origin of the map $f|_{V(L)}: X \cap V(L) \to \mathbb{C}$. But, it is easy to see that A'Campo's theorem implies that the Milnor fibre of $f|_{V(L)}$ cannot have the complex cohomology of a point.

As $f \in \mathfrak{m}^2_{X,\mathbf{0}}$, certainly $f|_{V(L)} \in \mathfrak{m}^2_{X \cap V(L),\mathbf{0}}$. Theorem 1.20 says then that the Lefschetz number of the monodromy isomorphism on the complex cohomology of $F_{f|_{V(L)},\mathbf{0}}$ is zero. If the cohomology were that of a point, then this Lefschetz number would have to be non-zero. □

The assumption in the corollary that $f \in \mathfrak{m}^2_{X,0}$ is necessary; otherwise, a trivial counterexample would consist of any non-zero linear form on affine space where we have declared the origin to be a stratum.

COROLLARY 3.11. *The complex link of an isolated complete intersection singularity never has the complex cohomology of a point.*

Proof. Suppose that $f_0, f_1, \ldots, f_k$ define an isolated complete intersesction singularity of dimension $n - k$ at the origin in $\mathbb{C}^{n+1}$. Since the origin is a singular point, we may assume that the f_i are ordered so that f_0 is in the square of the maximal ideal at the origin of $X := V(f_1, \ldots, f_k)$. But then, as $X \cap V(f_0)$ has the origin as an *isolated* singularity, X has at worst a one-dimensional singular set and $V(f_0)$ intersects this singular set in at most the origin. Since $f_0, f_1, \ldots, f_k$ define a complete intersection, it follows that $f_0|_X$ has a stratified isolated critical point at the origin. The desired conclusion now follows from the previous corollary. □

We originally claimed that the complex link of a general isolated singularity could not be contractible, using the same proof as above. It was pointed out to us by M. Tibăr that the proof fails in the general case; using the notation from the proof, the fact that $X \cap V(f_0)$ has an isolated singularity does not generally imply that $f_0|_X$ has a stratified isolated critical point. Specifically, if $V(f_0)$ contains an entire component of X, $f_0|_X$ does not have a stratified isolated critical point.

In fact, an example of D.M.Q. Mond shows that there are isolated singularities with contractible complex links.

In the case of an isolated complete intersection singularity, one would hope that Corollary 3.11 is already known from some result appearing in [28]. The only result we find in [28] which might imply Corollary 3.11 is the formula 5.11.c for the number of spheres in the homotopy-type of the complex link; however, this formula is an alternating sum, and it looks distinctly unpleasant to prove algebraically that it gives a non-zero result. On the other hand, one can obtain the conclusion of Corollary 3.11 fairly easily by using the deep result of Looijenga and Steenbrink [29] that the Milnor number is greater than or equal to the Tjurina number.

4. STRATIFIED NON-ISOLATED SINGULARITIES

We would like to analyse the vanishing cycles of functions with possibly non-isolated critical points in the same way that we dealt with the isolated case. Unfortunately, in the non-isolated situation, stratified Morse theory will give us contributions from boundary strata for, in the non-isolated case, the critical locus will intersect the boundary of arbitrarily small balls. This boundary Morse data is too difficult to deal with.

To get around this boundary Morse data problem, we use an approach based on that of Lê in [17]—we describe how the Milnor fibre of f is built up from the Milnor fibre of the restriction of f to a hypersurface slice. In [17], Lê uses slices by hyperplanes; we allow the slice itself to have an isolated singularity. By allowing these more general hypersurface slices, we recover the formula of Lê and Greuel as presented in 5.11.a of [28]; this is a result about isolated complete intersection singularities.

We must first prove a technical result which tells us that we use neighbourhoods of a certain form to define the Milnor fibre.

LEMMA 4.1. *Suppose that g is tractable for f with respect to the good stratification $\mathscr{S} = \{S_\alpha\}$ of X at $\mathbf{0}$.*

Then, for ε small, $\neq 0$, and $0 < \eta \ll \varepsilon$, we may use neighbourhoods of the form $B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta)$ to define the Milnor fibre of f up to homotopy (here, B_ε is the closed ε-ball and $\mathring{\mathbb{D}}_\eta$ is the open η-disc); more precisely, for $0 < |\xi| \ll \eta \ll \varepsilon \ll 1$, $B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\xi)$ is homotopy-equivalent to the Milnor fibre of f at the origin.

In addition, if $\mathbf{F}^\bullet$ is a bounded complex of sheaves on $X - V(f)$, constructible with respect to $\{S_\alpha \in \mathscr{S} \mid S_\alpha \nsubseteq V(f)\}$, then

$$\mathbb{H}^*(F_{f,\mathbf{0}}; \mathbf{F}^\bullet) \cong \mathbb{H}^*(B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\xi); \mathbf{F}^\bullet).$$

Moreover, for $0 < \nu \ll \varepsilon$, we may use neighbourhoods of the form $B_\varepsilon \cap f^{-1}(\mathring{\mathbb{D}}_\nu)$ to define the Milnor fibre of g up to homotopy or up to hypercohomology with coefficients in $\mathbf{F}^\bullet$ if $\mathbf{F}^\bullet$ is a bounded complex of sheaves on $X - V(g)$ which is constructible with respect to $\{S_\alpha - V(g) \mid S_\alpha \in \mathscr{S}\}$.

Proof. As we demonstrated in [33], to show that we may use neighbourhoods of the form $B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta)$ to define the Milnor fibre of f up to homotopy, what we must show is that for two such neighbourhoods

$$B_{\varepsilon_2} \cap g^{-1}(\mathring{\mathbb{D}}_{\eta_2}) \subseteq B_{\varepsilon_1} \cap g^{-1}(\mathring{\mathbb{D}}_{\eta_1})$$

for all ξ sufficiently small $\neq 0$, the inclusion

$$B_{\varepsilon_2} \cap g^{-1}(\mathring{\mathbb{D}}_{\eta_2}) \cap f^{-1}(\xi) \subseteq B_{\varepsilon_1} \cap g^{-1}(\mathring{\mathbb{D}}_{\eta_1}) \cap f^{-1}(\xi)$$

is a homotopy-equivalence. To prove the statement about hypercohomology, we must show that this previous inclusion induces an isomorphism on hypercohomology. We also need to show the analogous statements with f and g interchanged.

It suffices to show that, for $0 < \nu, \eta \ll a < \varepsilon$, the map

$$\begin{array}{c} X \cap \Psi^{-1}((a, \varepsilon) \times \mathring{\mathbb{D}}_\eta^* \times \mathring{\mathbb{D}}_\nu^*) \\ \downarrow \; {\scriptstyle \Psi := (|\mathbf{z}|^2, g, f)} \\ (a, \varepsilon) \times \mathring{\mathbb{D}}_\eta^* \times \mathring{\mathbb{D}}_\nu^* \end{array}$$

is a proper, stratified submersion. Recall that, as $\mathscr{S}$ is a good stratification for f at $\mathbf{0}$, the strata of $\mathscr{S}$ away from $f^{-1}(0)$ satisfy the Whitney conditions.

Let ε be so small that for all ε' such that $0 < \varepsilon' \leqslant \varepsilon$, $\partial B_{\varepsilon'}$ transversely intersects the strata $\{g^{-1}(0) \cap S_\alpha \mid S_\alpha \subseteq V(f)\}$ (we are using our hypothesis that g restricted to such an S_α has, at worst, an isolated critical point at the origin).

As the symmetric polar variety is one-dimensional, we may choose ε so small that $B_\varepsilon \cap \tilde{\Gamma}_{f,g} \cap V(g) \subseteq \{\mathbf{0}\}$, and so, for any a such that $0 < a < \varepsilon$, for all η sufficiently small,

$$B_\varepsilon \cap \Gamma_{f,g} \cap g^{-1}(\mathring{\mathbb{D}}_\eta^*) \subseteq B_a.$$

Another way of writing this is

$$\Psi^{-1}((a, \varepsilon) \times \mathring{\mathbb{D}}_\eta^* \times \mathbb{C}) \cap \Gamma_{f,g} = \emptyset.$$

Now, let $\tilde{g}$ be an extension of g to an open neighbourhood of the origin in $\mathbb{C}^N$, and suppose that no matter how small we pick η and ν, the map Ψ still has critical points. Then, there would exist a stratum $S_\alpha \nsubseteq V(f)$ and a sequence of points $\mathbf{p}_i \in S_\alpha - V(f) - V(g)$ such that the $\mathbf{p}_i$ converge to some point $\mathbf{p} \in V(f) \cap V(g) - \mathring{B}_a$ and such that

$$T_{\mathbf{p}_i} V(f|_{S_\alpha} - f(\mathbf{p}_i)) \cap T_{\mathbf{p}_i} V(\tilde{g} - \tilde{g}(\mathbf{p}_i)) \subseteq T_{\mathbf{p}_i} \partial B_{|\mathbf{p}|}.$$

(Here, we have used that $f \neq 0$ so that $\mathbf{p}_i \notin \Sigma(f|_{S_\alpha})$, that $g \neq 0$ so that $\mathbf{p}_i \notin \Sigma\tilde{g}$, and that the $\mathbf{p}_i$ are not in $\Gamma^1_{f,g}$ by the above.)

Let $S_\beta \subseteq V(f)$ be the stratum containing $\mathbf{p}$. We may assume that $T_{\mathbf{p}_i} V(\tilde{g} - \tilde{g}(\mathbf{p}_i))$ converges to some $\mathfrak{T}$ and that $T_{\mathbf{p}_i} V(f|_{S_\alpha} - f(\mathbf{p}_i))$ converges to some $\mathscr{T}$.

Then, we have that $T_{\mathbf{p}} S_\beta \subseteq \mathscr{T}$ by the good condition, $\mathfrak{T} + T_{\mathbf{p}} S_\beta = \mathbb{C}^N$ by prepolarity, and $\mathscr{T} \cap \mathfrak{T} \subseteq T_{\mathbf{p}} \partial B_{|\mathbf{p}|}$. But, by the choice of ε, $\mathfrak{T} \cap T_{\mathbf{p}} S_\beta + T_{\mathbf{p}} \partial B_{|\mathbf{p}|} = \mathbb{C}^N$; this is a contradiction. □

We now give our principal result of this section—a result which helps describe the hypercohomology of the Milnor fibre of a function by modding-out by the Milnor fibre of a "generic" hypersurface slice.

THEOREM 4.2. *Suppose that g is tractable relative to f with respect to a good stratification $\mathscr{S} = \{S_\alpha\}$ of X at $\mathbf{0}$. Let d_α denote the dimension of S_α.*

(A) *Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves of $\mathbb{Z}$-modules on $X - V(f)$, constructible with respect to $\{S_\alpha \in \mathscr{S} \mid S_\alpha \not\subseteq V(f)\}$.*

Then, for all i, $\mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet)$ is a direct summand of $H^{i-1}(\phi_g \psi_f \mathbf{F}^\bullet)_\mathbf{0}$, and there exist integers j_α such that

$$\mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet) \cong \bigoplus_{S_\alpha \not\subseteq V(f)} (\mathbb{H}^{i-d_\alpha+1}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))^{j_\alpha}$$

where $j_\alpha \geqslant (\tilde{\Gamma}^1_{f,g}(S_\alpha) \cdot V(f))_\mathbf{0}$ with equality if g is decent relative to f.

Furthermore, if $\Gamma_{f,g}(\mathscr{S})$ has no components contained in $V(g)$ (i.e. if $\Gamma_{f,g}(\mathscr{S}) = \tilde{\Gamma}^1_{f,g}(\mathscr{S})$), then

$$H^{i-1}(\phi_g \psi_f \mathbf{F}^\bullet)_\mathbf{0} \cong \mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet).$$

(B) *Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves of $\mathbb{Z}$-modules on $X - V(g)$, constructible with respect to $\{S_\alpha - V(g) \mid S_\alpha \in \mathscr{S}\}$.*

Then, for all i, $\mathbb{H}^i(F_{g,\mathbf{0}}, F_{g|_{V(f)},\mathbf{0}}; \mathbf{F}^\bullet)$ is a direct summand of $H^{i-1}(\phi_f \psi_g \mathbf{F}^\bullet)_\mathbf{0}$, and there exist integers r_α such that

$$\mathbb{H}^i(F_{g,\mathbf{0}}, F_{g|_{V(f)},\mathbf{0}}; \mathbf{F}^\bullet) \cong \bigoplus_{S_\alpha \not\subseteq V(f)} (\mathbb{H}^{i-d_\alpha+1}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))^{r_\alpha}$$

where $r_\alpha \geqslant (\tilde{\Gamma}^1_{f,g}(S_\alpha) \cdot V(g))_\mathbf{0}$ with equality if f is decent relative to g.

Furthermore, if $\hat{\mathscr{S}}$ is a Whitney stratification of X with respect to which f has a stratified isolated critical point at the origin such that $\mathscr{S}$ is the good stratification induced by f, and $\mathbf{F}^\bullet$ is constructible with respect to $\{S - V(g) \mid S \in \hat{\mathscr{S}}\}$, then

$$H^{i-1}(\phi_f \psi_g \mathbf{F}^\bullet)_\mathbf{0} \cong \mathbb{H}^i(F_{g,\mathbf{0}}, F_{g|_{V(f)},\mathbf{0}}; \mathbf{F}^\bullet).$$

Proof. This follows from Lemma 4.1, Lemma 2.2, and Theorem 3.2.

We first prove (A).

By Lemma 4.1, we may use neighbourhoods of the form $B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta)$ to define the Milnor fibre of f. It follows that

$$H^{i-1}(\phi_g \psi_f \mathbf{F}^\bullet)_\mathbf{0} \cong \mathbb{H}^i(B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\tau), B_\varepsilon \cap g^{-1}(\gamma) \cap f^{-1}(\tau); \mathbf{F}^\bullet)$$

where $0 < |\tau| \ll |\gamma| < \eta \ll \varepsilon \ll 1$. Now, consider the map $\tilde{g}$, the restriction of g,

$$\begin{gathered} B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\tau) \\ \downarrow \tilde{g} \\ \mathring{\mathbb{D}}_\eta. \end{gathered}$$

The space $B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\tau)$ is given the obvious induced stratification; this is a complex Whitney stratified set with boundary, in the sense of Definition 2.1. Note that the interior strata are of the form $S_\alpha \cap f^{-1}(\tau)$, which has dimension $d_\alpha - 1$. This -1 accounts for the $+1$ shift appearing in the degree of the hypercohomology of the normal Morse data in the statement of the theorem.

The map $\tilde{g}$ may have arbitrarily bad singularities over 0 but, away from $g^{-1}(0)$, all interior critical points are isolated since $\dim_{\mathbf{0}} \tilde{\Gamma}^1_{f,g}(\mathscr{S}) = 1$. Moreover, our proof of Lemma 4.1 shows that there are no boundary critical points except, possibly, over 0.

Let the possible stratified critical values of $\tilde{g}$ be $0, v_1, \ldots, v_m$. Let $\mathring{\mathbb{D}}_0, \mathring{\mathbb{D}}_1, \ldots, \mathring{\mathbb{D}}_m$ be a collection of disjoint discs in $\mathring{\mathbb{D}}_\eta$, centred at $0, v_1, \ldots, v_m$, respectively. For all i between 1 and m, let ξ_i be in $\mathring{\mathbb{D}}_i - \{v_i\}$. We will apply Lemma 2.2(A) to the map g restricted to the space $Y := B_\varepsilon \cap f^{-1}(\tau)$. For all sufficiently small $\xi \neq 0$

$$\begin{aligned} &\mathbb{H}^i(B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\tau), B_\varepsilon \cap g^{-1}(\gamma) \cap f^{-1}(\tau); \mathbf{F}^\bullet) \\ &\quad \cong \mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_0), Y \cap g^{-1}(\xi); \mathbf{F}^\bullet) \oplus \bigoplus_{i=1}^{m} \mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_i), Y \cap g^{-1}(\xi_i); \mathbf{F}^\bullet) \\ &\quad \cong \mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_0), Y \cap g^{-1}(\xi); \mathbf{F}^\bullet) \oplus \mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_\eta), Y \cap g^{-1}(0); \mathbf{F}^\bullet) \\ &\quad \cong \mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_0), Y \cap g^{-1}(\xi); \mathbf{F}^\bullet) \oplus \mathbb{H}^i(B_\varepsilon \cap g^{-1}(\mathring{\mathbb{D}}_\eta) \cap f^{-1}(\tau), B_\varepsilon \cap g^{-1}(0) \cap f^{-1}(\tau); \mathbf{F}^\bullet) \\ &\quad \cong \mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_0) \cap f^{-1}(\tau), Y \cap g^{-1}(\xi) \cap f^{-1}(\tau); \mathbf{F}^\bullet) \oplus \mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet) \end{aligned}$$

where the first ismorphism follows from Lemma 2.2(A), where we use ξ for the v in Lemma 2.2(A), and the second isomorphism also follows from Lemma 2.2(A), but using 0 for the v in Lemma 2.2(A). The last isomorphism follows from Lemma 4.1.

If $\Gamma^1_{f,g}(\mathscr{S})$ has no components contained in $V(g)$, then 0 is not a stratified critical value of $\tilde{g}$, and so

$$\mathbb{H}^i(Y \cap g^{-1}(\mathring{\mathbb{D}}_0) \cap f^{-1}(\tau), Y \cap g^{-1}(\xi) \cap f^{-1}(\tau); \mathbf{F}^\bullet) = 0.$$

Thus, in this case,

$$H^{i-1}(\phi_g \psi_f \mathbf{F}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet).$$

To finish (A), we must still rewrite $\mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet)$ in terms of the normal data to the various strata.

The map $\tilde{g}$ has, at worst, isolated stratified critical points over $\mathring{\mathbb{D}}_\eta - \{\mathbf{0}\}$. Hence, by the above homotopy argument, combined with the fact that local Morse data is Morse data, we conclude that $\mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet)$ is a direct sum of the hypercohomologies of the local Morse data at each of the isolated critical points of $\tilde{g}$ which do not lie above 0. Thus, it remains for us to show how these local contributions can be written in terms of the normal data to the strata.

But, at each critical point $\mathbf{p} \in S_\alpha$, we are in the situation of Theorem 3.2, and so we know that the relative hypercohomology splits as a direct sum of the shifted hypercohomologies of the normal data to the strata. Moreover, $(\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(f-f(\mathbf{p})))_{\mathbf{p}}$ is precisely the Milnor number at $\mathbf{p}$ of g restricted to $S_\alpha \cap V(f-f(\mathbf{p}))$, which is the number of complex Morse singularities which appear on S_α near $\mathbf{p}$ when g is perturbed slightly (recall the proof of Theorem 3.2). However, if the critical point, $\mathbf{p}$, is not decent, then—when we perturb g near $\mathbf{p}$—extra critical points can arise near $\mathbf{p}$ on strata which contain S_α in their closure. (A) follows.

The proof of (B) is the same as that of (A), except for the conditions under which we can conclude that

$$H^{i-1}(\phi_f\psi_g\mathbf{F}^\bullet)_{\mathbf{0}} \cong \mathbb{H}^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbf{F}^\bullet).$$

As in the proof of (A), but with f and g interchanged, we need to know that 0 is not a stratified critical value of f restricted to $g^{-1}(\tau)$ inside B_ε. If $\mathscr{S}$ is the stratification induced from f having a stratified isolated critical point with respect to the stratification $\hat{\mathscr{S}}$, then—since we are supposing that g is tractable with respect to $\mathscr{S}$—the pair of functions (f, g) has no stratified critical points on $f^{-1}(0)-\{\mathbf{0}\}$ with respect to $\hat{\mathscr{S}}$ in a small neighbourhood of the origin. The desired conclusion follows. □

Remark 4.3. Theorem 4.2 produces especially interesting results when $\mathbf{F}^\bullet|_{X-V(f)}$ is positively perverse, for then Remark 1.19 tells us that $\mathbb{H}^{i-d_\alpha+1}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet)$ can only be non-zero if $i = \dim X - 1$.

Also, note that in the proof of Theorem 4.2 we cannot conclude (B) above directly from (A) simply by interchanging f and g, for in both parts the assumption is that g is tractable for f. Note also that the tractibility of g prevents any non-zero strata from being contained in $V(g)$. One final note: though we state the theorem for complexes which need only be defined on the complements of $V(f)$ and $V(g)$, in most applications the complex $\mathbf{F}^\bullet$ will be defined on all of X.

COROLLARY 4.4. *Suppose that g is prepolar and decent relative to f with respect to a good stratification $\mathscr{S}=\{S_\alpha\}$ of X at $\mathbf{0}$. Suppose also that f is decent relative to g. Let s denote the dimension of X.*

Then,

$$\chi(\phi_g\phi_f\mathbf{F}^\bullet)_{\mathbf{0}} + \chi(\phi_g\mathbf{F}^\bullet)_{\mathbf{0}} = (-1)^s \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)((\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(f))_{\mathbf{0}} - (\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(g))_{\mathbf{0}})$$

where $m_\alpha(\mathbf{F}^\bullet)$ is the normal index of S_α with respect to $\mathbf{F}^\bullet$.

Proof. Let d_α denote the dimension of S_α.

From the distinguished triangle relating the neighbouring and vanishing cycles along f, we have

$$\chi(\phi_g\phi_f\mathbf{F}^\bullet)_{\mathbf{0}} = \chi(\phi_g\psi_f\mathbf{F}^\bullet)_{\mathbf{0}} - \chi(\phi|_{g_{V(f)}}(\mathbf{F}^\bullet|_{V(f)}))_{\mathbf{0}}.$$

We can rewrite this last quantity:

$$\begin{aligned} -\chi(\phi_{g|_{V(f)}}(\mathbf{F}^\bullet|_{V(f)}))_{\mathbf{0}} &= \chi(\mathbb{H}^*(B_\varepsilon \cap V(f), F_{g|_{V(f)},\mathbf{0}}; \mathbf{F}^\bullet)) \\ &= \chi(\mathbb{H}^i(B_\varepsilon, F_{g,\mathbf{0}}; \mathbf{F}^\bullet)) + \chi(\mathbb{H}^*(F_{g,\mathbf{0}}, F_{g|_{V(f)},\mathbf{0}}; \mathbf{F}^\bullet)) \\ &= -\chi(\phi_g\mathbf{F}^\bullet)_{\mathbf{0}} + \chi(\mathbb{H}^*(F_{g,\mathbf{0}}, F_{g|_{V(f)},\mathbf{0}}; \mathbf{F}^\bullet)). \end{aligned}$$

Combining this with our previous equation and applying both parts of Theorem 4.2, we find that

$$\chi(\phi_g\phi_f\mathbf{F}^\bullet)_0 = -\chi(\phi_g\mathbf{F}^\bullet)_0 - \sum_{S_\alpha \not\subseteq V(f)} (-1)^{d_\alpha - 1}\chi(\mathbb{H}^*(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))\,((\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(g))_0).$$

The corollary now follows from the definition of the normal indices $m_\alpha(\mathbf{F}^\bullet)$. □

We can now prove the exponent statement made in Theorem 3.2 and the claims made in Remark 3.4 regarding how generic the linear form L must be.

Corollary 4.5. *Let $\mathcal{S}$ be a Whitney stratification for X, let $\mathbf{F}^\bullet$ be a bounded complex of sheaves which is constructible with respect to $\mathcal{S}$, and suppose that $f: X \to \mathbb{C}$ has a stratified isolated singularity at the origin. Let $\mathcal{T}$ denote the good stratification induced by f.*

If $g: X \to \mathbb{C}$ is prepolar with respect to $\mathcal{T}$, g is decent with respect to $\mathcal{S}$ relative to f, f is decent with respect to $\mathcal{S}$ relative to g, and $d_0 g$ is non-degenerate with respect to $\mathcal{S}$, then for all i

$$H^{i-1}(\phi_f\mathbf{F}^\bullet)_0 \cong \mathbb{H}^i(B_\varepsilon \cap X, F_{f,0}; \mathbf{F}^\bullet) \cong \mathbb{H}^i(B_\varepsilon \cap X, \mathbb{L}_{X,0}; \mathbf{F}^\bullet) \oplus \bigoplus_{S_\alpha \not\subseteq V(f)} (\mathbb{H}^{i-d_\alpha}(c(\mathbb{L}_\alpha), \mathbb{L}_\alpha; \mathbf{F}^\bullet))^{k_\alpha}$$

where

$$k_\alpha = (\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(g))_0.$$

Proof. The only portion of this statement that we did not prove in Theorem 3.2 is the description of the k_α for $S_\alpha \neq \{\mathbf{0}\}$.

Let $s = \dim X$. In the proof of Theorem 3.2, we did show that $k_\gamma = (-1)^{s-1}\chi(\phi_f\mathbf{U}^\bullet(X, S_\gamma; 1))_0$; we will show that this equality, combined with Corollary 4.4, yields the desired result.

As the origin is a stratified isolated critical point of f, the origin is an isolated point of the support of the cohomology sheaves $\mathbf{H}(\phi_f\mathbf{F}^\bullet)$ (recall Proposition 1.16). Therefore, $H^*(\psi_g\phi_f\mathbf{F}^\bullet)_0 = 0$ and so, from the distinguished triangle relating the neighbouring and vanishing cycles, we have that $\chi(\phi_g\phi_f\mathbf{F}^\bullet)_0 = -\chi(\phi_f\mathbf{F}^\bullet)_0$. Also, as $d_0 g$ is non-degenerate, $(-1)^{s-1}\chi(\phi_g\mathbf{F}^\bullet)_0 = m_{\{0\}}(\mathbf{F}^\bullet) =$ the normal index of $\mathbf{F}^\bullet$ at the origin.

Now, applying Corollary 4.4, we find

$$\begin{aligned} k_\gamma &= (-1)^{s-1}\chi(\phi_f\mathbf{U}^\bullet(X, S_\gamma; 1))_0 \\ &= m_{\{0\}}(\mathbf{U}^\bullet(X, S_\gamma; 1)) + \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{U}^\bullet(X, S_\gamma; 1))(\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,g}(S_\alpha)\cdot V(g))_0. \end{aligned}$$

But now, the definition of $\mathbf{U}^\bullet(X, S_\gamma; 1)$ implies that

$$k_\gamma = (\tilde{\Gamma}^1_{f,g}(S_\gamma)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,g}(S_\gamma)\cdot V(g))_0.$$

and we are finished. □

Corollary 4.6. *Let X be an s-dimensional analytic subspace of an open subset of some affine space, and suppose $\mathbf{0} \in X$. Let $f: (X, \mathbf{0}) \to (\mathbb{C}, 0)$ be an analytic function, and let $\mathcal{S} = \{S_\alpha\}$ be a good stratification for f at $\mathbf{0}$ which includes $\{\mathbf{0}\}$ as a stratum. Let $\mathbf{F}^\bullet$ be a bounded complex of sheaves of $\mathbb{Z}$-modules on X which is constructible with respect to $\mathcal{S}$.*

Then, the normal indices at the origin of $\phi_f \mathbf{F}^\bullet$, $\psi_f \mathbf{F}^\bullet$, and $\mathbf{F}^\bullet|_{V(f)}$ can be computed by the following formula:

$$m_0(\psi_f \mathbf{F}^\bullet) = \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0$$

$$m_0(\mathbf{F}^\bullet) + m_0(\mathbf{F}^\bullet|_{V(f)}) = \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0$$

and

$$m_0(\phi_f \mathbf{F}^\bullet) = m_0(\mathbf{F}^\bullet) + \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)((\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0),$$

where L is a generic linear form.

Proof. The formula for $m_0(\psi_f \mathbf{F}^\bullet)$ follows at once from Theorem 4.2 (A) by setting g equal to a generic linear form L. The formula for $m_0(\phi_f \mathbf{F}^\bullet)$ follows from Corollary 4.4 by once again setting g equal to a generic linear L. The remaining formula follows from the other two since the characteristic cycle $Ch(\mathbf{F}^\bullet|_{V(f)})$ can be computed via the formula $Ch(\mathbf{F}^\bullet|_{V(f)}) = Ch(\psi_f \mathbf{F}^\bullet) - Ch(\phi_f \mathbf{F}^\bullet)$. □

5. APPLICATIONS

Example 5.1. We recover the result of Lê in [17] by applying Theorem 4.2 (A) to the case where X is an open neighbourhood of the origin in $\mathbb{C}^N, f$ is any analytic function, g is a generic linear form, and $\mathbf{F}^\bullet = \mathbb{Z}^\bullet_X$. The precise condition that we need on the linear form is that it is tractable relative to f at the origin; in our present setting, this equivalent to the form being prepolar.

The only stratum of X not contained in $V(f)$ is $X - V(f)$, which has normal data (*point*, $\emptyset$). Decency is trivially satisfied.

Thus, the conclusion of Theorem 4.2(A) is that $H^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbb{Z})$ is zero except in degree $N-1$ and $H^{N-1}(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbb{Z})$ is free Abelian of rank $(\tilde{\Gamma}^1_{f,g}\cdot V(f))_0$, where we have written $\tilde{\Gamma}^1_{f,g}$ in place of $\tilde{\Gamma}^1_{f,g}(X - V(f)) = \tilde{\Gamma}^1_{f,g}(X)$.

Example 5.2. Let X^s be a purely s-dimensional space which has an isolated singularity at the origin. Let X_i denote the irreducible components of X at the origin. Suppose that we have an analytic map $f:(X, \mathbf{0}) \to (\mathbb{C}, 0)$. Let $\mathscr{S} = \{S_\alpha\}$ be a good stratification of X for f at the origin which has all of the $X_i - V(f)$ as strata. Suppose that $g:(X,\mathbf{0}) \to (\mathbb{C}, 0)$ is tractable with respect to $\mathscr{S}$ (we are *not* assuming that g is linear). We will once again use constant coefficients.

The normal Morse data to each of $X_i - V(f)$ is just (*point*, $\emptyset$). In addition, the decency of g is trivial. Thus, Theorem 4.2 (A) tells us that $H^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbb{Z})$ is zero except in degree $s-1$ and $H^{s-1}(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbb{Z})$ is free Abelian of rank

$$\sum_i (\tilde{\Gamma}^1_{f,g}(X_i)\cdot V(f))_0 = (\tilde{\Gamma}^1_{f,g}(\mathscr{S})\cdot V(f))_0.$$

This concentration of the relative cohomology in degree $s-1$ is what makes the approach of studying $F_{f,\mathbf{0}}$ by taking hypersurface slices so appealing. In the more general setting where we have coefficients in a complex $\mathbf{F}^\bullet$, Remark 4.3 tells us that the relative hypercohomology will still be concentrated in degree $s-1$ provided that $\mathbf{F}^\bullet|_{X-V(f)}$ is positively perverse. In the situation above, $X - V(f)$ is a smooth manifold and so the constant sheaf $\mathbb{Z}^\bullet|_{X-V(f)}$ is positively perverse.

Of course, the question remains: how does one calculate the symmetric polar curve in this case? Suppose that X is defined as a reduced subspace of some open neighbourhood of the origin in $\mathbb{C}^N$ by the vanishing of $h_1, \dots, h_l$. Then, as in Remark 1.7, the symmetric polar curve $\tilde{\Gamma}^1_{f,g}(\mathscr{S})$ is defined as a cycle by beginning with the cycle defined by the subscheme $V(h_1, \dots, h_l, J_{N-s+2}(h_1, \dots, h_l, f, g))$ and then discarding any components which are contained in $V(f)$ or $V(g)$.

This is related to results in [3].

Example 5.3. A particular case of the above example is of special interest; the case where X is purely s-dimensional and has an isolated singularity at the origin, and f itself has, at worst, an isolated critical point at the origin. Then, we have the induced good stratification of X for f given by

$$\{X_i - V(f), X_i \cap V(f) - \mathbf{0}, \mathbf{0} \mid X_i \textit{ a component of } X\}.$$

A function $g: X \to \mathbb{C}$ is tractable with respect to this induced stratification if and only if, for all components X_i of X, the critical locus of $(f, g)|_{X_i}$ is at most one-dimensional at the origin with no one-dimensional components contained in $V(f)$ or $V(g)$.

When g is tractable, we conclude from the previous example that $H^i(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbb{Z})$ is zero except in degree $s-1$ and $H^{s-1}(F_{f,\mathbf{0}}, F_{f|_{V(g)},\mathbf{0}}; \mathbb{Z})$ is free Abelian of rank

$$(V(h_1, \dots, h_l, J_{N-s+2}(h_1, \dots, h_l, f, g)) \cdot V(f))_{\mathbf{0}}$$

where, as before, X is defined with its reduced structure by $V(h_1, \dots, h_l)$ in some open neighbourhood of the origin in $\mathbb{C}^N$. This is a generalization of the formula of Lê and Greuel (see [17, 21,13, 28, 5.11.a]) for the Milnor number of an isolated complete intersection.

To obtain Lê and Greuel's precise result, we must suppose that X is a local complete intersection. Then, as the constant sheaf on a local complete intersection is positively perverse, it follows from 1.16 (or see [20, 22, 23]) that $F_{f,\mathbf{0}}$ and $F_{f_{|V(g)},\mathbf{0}}$ have non-trivial reduced cohomology only in dimensions $s-1$ and $s-2$, respectively. Also, one may have to replace the defining equations in Lê and Greuel's formula with generic linear combinations to guarantee that the Milnor fibres in Lê and Greuel's sense (as in [28]) agree with the Milnor fibres that we are using. Finally, we may replace the intersection number above with simply

$$\dim_{\mathbb{C}} \mathscr{O}^N_{\mathbf{0}} / \langle J_{N-s+2}(h_1, \dots, h_l, f, g), h_1, \dots, h_l, f \rangle \mathscr{O}^N_{\mathbf{0}}$$

since

$$\mathscr{O}^N_{\mathbf{0}} / \langle J_{N-s+2}(h_1, \dots, h_l, f, g), h_1, \dots, h_l, f \rangle \mathscr{O}^N_{\mathbf{0}}$$

is Cohen–Macaulay by [28, 4.4].

Example 5.4. Here, we will consider an important special case of one of the formulas of Corollary 4.6:

$$m_{\mathbf{0}}(\mathbf{F}^\bullet) + m_{\mathbf{0}}(\mathbf{F}^\bullet|_{V(f)}) = \sum_{S_\alpha \not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)(\tilde{\Gamma}^1_{f,L}(S_\alpha) \cdot V(L))_{\mathbf{0}}.$$

Suppose that X^s is a local complete intersection with a singular set of arbitrary dimension. Then, the constant sheaf on X is positively perverse and the complex link of X at any point has the homotopy-type of a bouquet of $s-1$-spheres (again, see [20, 22, 23]). If $f:(X,\mathbf{0})\to(\mathbb{C},0)$ is such that $\dim_{\mathbf{0}} X\cap V(f)=s-1$, then $X'=X\cap V(f)$ is also a local complete intersection and the formula above becomes

$$b_{s-1}(\mathbb{L}_{X,\mathbf{0}})+b_{s-2}(\mathbb{L}_{X\cap V(f),\mathbf{0}})=\sum_\alpha m_\alpha(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_{\mathbf{0}}$$

where $b_i(\ )$ denotes the Betti number (with constant coefficients), $\mathcal{S}=\{S_\alpha\}$ is a good, Whitney stratification of X for f at the origin, L is a generic linear form, and m_α is the normal index of S_α relative to the constant sheaf.

In particular, if X has, at worst, an isolated singularity at the origin, then we obtain

$$b_{s-1}(\mathbb{L}_{X,0})+b_{s-2}(\mathbb{L}_{X\cap V(f),\mathbf{0}})=\sum_i(\tilde{\Gamma}^1_{f,L}(X_i)\cdot V(L))_{\mathbf{0}}$$

where the X_i are the irreducible components of X.

This generalizes the well-known formula from the affine hypersurface case, i.e. the case where $X=\mathbb{C}^s$; in this case the formula reduces to

$$b_{s-2}(\mathbb{L}_{V(f),\mathbf{0}})=(\tilde{\Gamma}^1_{f,L}\cdot V(L))_{\mathbf{0}}.$$

Example 5.5. We wish to look again at one at one of the formulas from Corollary 4.6:

$$m_0(\phi_f\mathbf{F}^\bullet)=m_0(\mathbf{F}^\bullet)+\sum_{S_\alpha\not\subseteq V(f)} m_\alpha(\mathbf{F}^\bullet)((\tilde{\Gamma}^1_{f,L}(S_\alpha\cdot V(f))_0-(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0).$$

On the level of Euler characteristics, this formula generalizes that of Theorem 3.2 by allowing arbitrary f. However, in this general case where the generalized critical locus of f has arbitrary dimension, we do not obtain the direct sum decomposition of the stalk cohomology of the vanishing cycles that we had in Theorem 3.2.

As a quick example, let us look at Example 1.8; in that example, f had a one-dimensional critical locus. Then, leaving the details as an exercise, one finds

$$m_{\{0\}}(\phi_f\mathbb{Z}^\bullet_X)=m_{\{0\}}(\mathbb{Z}^\bullet_X)+((\tilde{\Gamma}^1_{f,y}(X)\cdot V(f))_0-(\tilde{\Gamma}^1_{f,y}(X)\cdot V(y))_0)=1+(6-2)=5.$$

Example 5.6. Our final example will show how the formulas of Corollary 4.6 really do allow one to calculate the normal indices—and, hence, the characteristic cycles—of a number of important complexes of sheaves associated to a local complete intersection.

Consider Example 1 from 1.B of [28], where one has a codimension 2 complete intersection, X, in $\mathbb{C}^N$ defined by

$$f_1(z_1,\ldots,z_N)=z_1^2+\cdots+z_N^2$$

and

$$f_2(z_1,\ldots,z_N)=\lambda_1z_1^2+\cdots+\lambda_Nz_N^2$$

where $N\geqslant 2$. The complete intersection X has an isolated singularity at the origin if and only if all of the λ_i are distinct.

For our example, we will consider the easiest case where the singularity of X at the origin is *non-isolated*; we let $N=4$ and let exactly two of the λ_i be the same. We assume without loss of generality that $\lambda_1=\lambda_2$. After dividing through by the duplicated λ_i and subtracting f_1, we have reduced our example to the complete intersection X in $\mathbb{C}^4$ which is defined by

$$f(w,x,y,z)=w^2+x^2+y^2+z^2$$

and

$$g(w, x, y, z) = \gamma y^2 + \delta z^2$$

where $\gamma, \delta \neq 0$ and $\gamma \neq \delta$. One easily verifies that the space $X = V(f, g)$ has a one-dimensional singular set which is $\Sigma X = V(w^2 + x^2, y, z)$.

Let $M := \mathbb{C}^4$. We will calculate the normal indices at the origin, m_0, for a number of important complexes: $\mathbb{C}^\bullet_{V(f)}$, $\psi_f \mathbb{C}^\bullet_M$, $\phi_f \mathbb{C}^\bullet_M$, $\mathbb{C}^\bullet_X$, $\phi_g \mathbb{C}^\bullet_{V(f)}$ and $\phi_g \psi_f \mathbb{C}^\bullet_M$.

The calculations of the normal indices at the origin for $\mathbb{C}^\bullet_{V(f)}$, $\psi_f \mathbb{C}^\bullet_M$, and $\phi_f \mathbb{C}^\bullet_M$ are nothing new, since they only involve a single function on affine space. Hence, we will proceed fairly quickly here.

As $f: M \to \mathbb{C}$ has an isolated critical point at the origin, the complex $\phi_f \mathbb{C}^\bullet_M$ is supported only at the origin and only in degree 3. Moreover, in degree 3, the dimension of the cohomology is simply the Milnor number of f at $\mathbf{0}$, i.e. $\mu_\mathbf{0}(f)$. Hence $m_0(\phi_f \mathbb{C}^\bullet_M) = \mu_\mathbf{0}(f) = 1$.

The second formula of Corollary 4.6 reduces in our current setting to the well-known (though, perhaps, unusually stated) $m_\mathbf{0}(\mathbb{C}^\bullet_{V(f)}) = (\Gamma^1_{f,L} \cdot V(L))_\mathbf{0}$, where L is a generic linear form and $\Gamma^1_{f,L}$ is the ordinary relative polar curve of Lê and Teissier (see also Example 5.4).

Let $L = aw + bx + cy + dz$, where we need to choose (a, b, c, d) generically. Then, $\Gamma^1_{f,L}$ is simply the line which is described parametrically by $(w, x, y, z) = t(a, b, c, d)$, provided that $a^2 + b^2 + c^2 + d^2 \neq 0$. Therefore, we find

$$m_\mathbf{0}(\mathbb{C}^\bullet_{V(f)}) = (\Gamma^1_{f,L} \cdot V(L))_\mathbf{0} = 1.$$

As there is the fundamental distinguished triangle

$$0 \to \mathbb{C}^\bullet_{V(f)} \to \psi_f \mathbb{C}^\bullet_M \to \phi_f \mathbb{C}^\bullet_M \to 0$$

and since characteristic cycles are additive over distinguished triangles, we deduce from the two calculations above that

$$m_\mathbf{0}(\psi_f \mathbb{C}^\bullet_M) = m_\mathbf{0}(\phi_f \mathbb{C}^\bullet_M) + m_\mathbf{0}(\mathbb{C}^\bullet_{V(f)}) = 2.$$

Of course, if we want the entire characteristic cycles of $\mathbb{C}^\bullet_{V(f)}$, $\psi_f \mathbb{C}^\bullet_M$, and $\phi_f \mathbb{C}^\bullet_M$, we have a little more to write; each of these complexes is constructible with respect to the obvious Whitney stratification of $V(f)$ given by $\mathscr{S} := \{V(f) - \mathbf{0}, \mathbf{0}\}$. We will write $m_{V(f)}$ for the normal index to the stratum $V(f) - \mathbf{0}$.

Then, clearly $m_{V(f)}(\mathbb{C}^\bullet_{V(f)}) = 1$ and $m_{V(f)}(\phi_f \mathbb{C}^\bullet_M) = 0$. Thus, as above, $m_{V(f)}(\psi_f \mathbb{C}^\bullet_M) = m_{V(f)}(\mathbb{C}^\bullet_{V(f)}) + m_{V(f)}(\phi_f \mathbb{C}^\bullet_M) = 1$.

Hence, the characteristic cycles are

$$Ch(\mathbb{C}^\bullet_{V(f)}) = [T^*_\mathbf{0} M] + [T^*_{V(f)} M]$$

$$Ch(\phi_f \mathbb{C}^\bullet_M) = [T^*_\mathbf{0} M]$$

and

$$Ch(\psi_f \mathbb{C}^\bullet_M) = 2[T^*_\mathbf{0} M] + [T^*_{V(f)} M]$$

where we have written $T^*_{V(f)} M$ for the closure in T^*M of $T^*_{V(f)-\mathbf{0}} M$.

With the characteristics cycles above in hand, we are now ready to calculate the normal indices at the origin for the more complicated complexes on $X = V(f, g)$: the complexes $\mathbb{C}^\bullet_X$, $\phi_g \mathbb{C}^\bullet_{V(f)}$, and $\phi_g \psi_f \mathbb{C}^\bullet_M$. In each of these situations, we are considering g as a function restricted to $V(f)$, where $V(f)$ is endowed with the stratification $\mathscr{S}$ above.

Looking at Corollary 4.6, we see that the fundamental problem in calculating the normal indices that we want is the problem of finding the symmetric polar curve $\tilde{\Gamma}^1_{g,L}(V(f)-\mathbf{0})$ for a generic choice of the linear form L. By Proposition 1.12, for generic L, $\tilde{\Gamma}^1_{g,L}(V(f)-\mathbf{0}) = \Gamma^1_{g,L}(V(f)-\mathbf{0})$. Throughout the rest of this example, we shall write simply $\Gamma^1_{g,L}(V(f))$ in place of $\Gamma^1_{g,L}(V(f)-\mathbf{0})$.

Let $L = aw + bx + cy + dz$. From Remark 1.7, $\Gamma^1_{g,L}(V(f))$ is obtained by considering the cycle defined by $V(f, J_3(f, g, L))$ and then discarding any components contained in $V(g)$. One easily calculates that

$$V(f, J_3(f, g, L)) = V(w^2 + x^2 + y^2 + z^2, y(ax - bw), z(ax - bw),$$

$$w(d\gamma y - c\delta z) + a(\delta - \gamma)yz, x(d\gamma y - c\delta z) + b(\delta - \gamma)yz)$$

where we have used that $\gamma, \delta \neq 0$. Unfortunately, $V(f, J_3(f, g, L))$ certainly has components contained in $V(g)$; in fact, $\Sigma X = V(w^2 + x^2, y, z) \subseteq V(f, J_3(f, g, L))$. We must dispose of these bad components.

We alter the generators of $V(f, J_3(f, g, L))$ slightly by subtracting the third generator from the second and writing the variety as

$$V(w^2 + x^2 + y^2 + z^2, (y - z)(ax - bw), z(ax - bw),$$

$$w(d\gamma y - c\delta z) + a(\delta - \gamma)yz, x(d\gamma y - c\delta z) + b(\delta - \gamma)yz).$$

This description of $V(f, J_3(f, g, L))$ has the advantage that if one considers only the $(y - z)$ factor in the second generator and leaves all the other generators the same, then one obtains

$$V(w^2 + x^2 + y^2 + z^2, y - z, z(ax - bw),$$

$$w(d\gamma y - c\delta z) + a(\delta - \gamma)yz, x(d\gamma y - c\delta z) + b(\delta - \gamma)yz)$$

which is completely contained in the set $V(g) = V(\gamma y^2 + \delta z^2)$ for generic (a, b, c, d) (specifically, we need to have $(a^2 + b^2)(\delta - \gamma)^2 + 2(d\gamma - c\delta)^2 \neq 0$).

Since $\Gamma^1_{g,L}(V(f))$ is obtained from $V(f, J_3(f, g, L))$ by disposing of components contained in $V(g)$, the last paragraph tells us that we may throw away those components coming from the $(y - z)$ factor in the second generator. Hence, $\Gamma^1_{g,L}(V(f))$ can be obtained from

$$V(w^2 + x^2 + y^2 + z^2, ax - bw, z(ax - bw), w(d\gamma y - c\delta z) + a(\delta - \gamma)yz,$$

$$x(d\gamma y - c\delta z) + b(\delta - \gamma)yz)$$

by throwing away any components still remaining in $V(g)$.

But this last variety is easily seen to equal

$$V(w^2 + x^2 + y^2 + z^2, ax - bw, w(d\gamma y - c\delta z) + a(\delta - \gamma)yz)$$

if $a \neq 0$, and then one sees that there are no components remaining in $V(g)$. Therefore, we finally arrive at an equality of cycles

$$\Gamma^1_{g,L}(V(f)) = V(w^2 + x^2 + y^2 + z^2, ax - bw, w(d\gamma y - c\delta z) + a(\delta - \gamma)yz).$$

Since $V(w^2 + x^2 + y^2 + z^2, ax - bw, w(d\gamma y - c\delta z) + a(\delta - \gamma)yz)$ is a complete intersection defined by homogeneous polynomials, and since g and L are each homogeneous, it is trivial to calculate the intersection numbers $(\Gamma^1_{g,L}(V(f)) \cdot V(g))_{\mathbf{0}}$ and $(\Gamma^1_{g,L}(V(f)) \cdot V(L))_{\mathbf{0}}$—one simply multiplies the multiplicities of all of the polynomials together. Therefore, we find $(\Gamma^1_{g,L}(V(f)) \cdot V(g))_{\mathbf{0}} = 8$ and $(\Gamma^1_{g,L}(V(f)) \cdot V(L))_{\mathbf{0}} = 4$.

At last, we are in a position to apply the last two formulas of Corollary 4.6 to find:

$$m_0(\mathbb{C}^\bullet_{V(f)}) + m_0(\mathbb{C}^\bullet_{|V(f,g)}) = m_{V(f)}(\mathbb{C}^\bullet_{V(f)})(\Gamma^1_{g,L}(V(f))\cdot V(L))_0$$

$$m_0(\phi_g\mathbb{C}^\bullet_{V(f)}) = m_0(\mathbb{C}^\bullet_{V(f)}) + m_{V(f)}(\mathbb{C}^\bullet_{V(f)})((\Gamma^1_{g,L}(V(f))\cdot V(g))_0 - (\Gamma^1_{g,L}(V(f))\cdot V(L))_0)$$

$$m_0(\phi_g\psi_f\mathbb{C}^\bullet_M) = m_0(\psi_f\mathbb{C}^\bullet_M) + m_{V(f)}(\psi_f\mathbb{C}^\bullet_M)((\Gamma^1_{g,L}(V(f))\cdot V(g))_0 - (\Gamma^1_{g,L}(V(f))\cdot V(L))_0).$$

Combining these with our earlier calculations, we find:

$$m_0(\mathbb{C}^\bullet_X) = 3, \quad m_0(\phi_g\mathbb{C}^\bullet_{V(f)}) = 5 \quad \text{and} \quad m_0(\phi_g\psi_f\mathbb{C}^\bullet_M) = 6.$$

As before, to calculate the entire characteristic cycles of $\mathbb{C}^\bullet_X$, $\phi_g\mathbb{C}^\bullet_{V(f)}$, and $\phi_g\psi_f\mathbb{C}^\bullet_M$, there is more work to be done.

Certainly, the normal index to the smooth stratum for $\mathbb{C}^\bullet_X$ is equal to 1; however, some calculation must be done for the normal indices to the one-dimensional singular strata comprising $\Sigma X - \mathbf{0}$. In addition, $\phi_g\mathbb{C}^\bullet_{V(f)}$ and $\phi_g\psi_f\mathbb{C}^\bullet_M$ are supported on ΣX and so we need the normal indices to the strata comprising $\Sigma X - \mathbf{0}$ for these complexes also.

These indices are calculated by taking a normal slice at a point on each stratum—this reduces one to the isolated point-stratum case, which can be handled via Corollary 4.6. In our present example, this leads to a fairly trivial situation, for at points of $\Sigma X - \mathbf{0}$, $V(f)$ itself is smooth and so a normal slice reduces one to the case of an isolated affine hypersurface singularity (after an analytic change of coordinates). As this requires none of the machinery of this paper, we leave these remaining calculations as an exercise for the reader.

6. QUESTIONS AND REMARKS

One may well ask: why prove all of the results of this paper using the generality of hypercohomology? We have several reasons for this.

First, it is important to be able to use intersection cohomology coefficients. We did not give a complicated example of this, for such an example would occupy many pages all by itself.

Secondly, hypercohomology is useful for inductive applications of our results. For instance, Corollary 4.4 concludes something about the iterated vanishing cycles $\phi_g\phi_f\mathbf{F}^\bullet$. Even if $\mathbf{F}^\bullet$ is the constant sheaf, the stalk cohomology of $\phi_g\phi_f\mathbf{F}^\bullet$ is not easily expressible in terms of ordinary relative cohomology. In particular, the normal indices of the sheaves of nearby cycles and vanishing cycles seem to be important algebraic data to associate to an analytic function.

Finally, it takes no extra effort to prove the statements with complexes of coefficients. In fact, our use of Lemma 3.1 in the proofs of Theorem 3.2 and Corollary 4.5 makes it easier to do the general case than the constant coefficient case.

We have made a special point throughout this paper not to write

$$(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f) - V(L)))_0$$

in place of

$$(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0.$$

This is because we believe that the former expression would lead one to believe that the cycle $V(f) - V(L)$ is somehow important—this is not the case.

Consider, for instance, the affine hypersurface case; in this case, we know that

$$(\tilde{\Gamma}^1_{f,L}\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}\cdot V(L))_0 = (\tilde{\Gamma}^1_{f,L}\cdot(\partial f/\partial L))_0$$

and the cycle $V(\partial f/\partial L)$ is the important thing.

In the general case, we would also like to replace the $V(f) - V(L)$ with a single non-negative cycle, as one can do in the affine hypersurface case. At least one advantage to such a presentation would be to make it clear that this intersection number is non-negative; that this is true for generic L follows from [39], which tells us that for L generic enough, $\tilde{\Gamma}^1_{f,L}(S_\alpha)$ is reduced and $(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0$ is actually the multiplicity of $\tilde{\Gamma}^1_{f,L}(S_\alpha)$ at the origin.

Our proof of what the exponents k_α are in Theorem 3.2 is extremely roundabout. It seems as though it should be easy to prove directly that the number non-degenerate critical points which occur on each stratum in the proof of Theorem 3.2 is given by

$$(\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(f))_0 - (\tilde{\Gamma}^1_{f,L}(S_\alpha)\cdot V(L))_0.$$

We could find no such proof. Such a proof would not significantly reduce the length of this paper since the determination of the k_α follows quickly once one has Theorem 4.2. However, it would be aesthetically pleasing to know that a direct calculation of the k_α is possible. Also, such a calculation might provide an answer to our previous question of how to replace $V(f) - V(L)$ with a single non-negative cycle.

Finally, we wish to indicate where this work is leading.

Suppose $\mathcal{U}$ is an open neighbourhood of the origin in $\mathbb{C}^{n+1}$ and that we have a family of analytic functions $f_t : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ such that, for all t, $\dim_0 \Sigma f_t = 0$, i.e. we have a family of isolated affine hypersurface singularities. Then, if the Milnor number of f_t at the origin, $\mu_0(f_t)$, is independent of t, the homotopy-type of the Milnor fibre of f_t at the origin is also independent of t.

This last statement is tautologically true if one's definition of $\mu_0(f_t)$ is that it equals the number of spheres which are wedged together in the homotopy-type of the Milnor fibre of f_t at the origin. However, if one defines the Milnor number as

$$\mu_0(f_t) = \dim_{\mathbb{C}} \frac{\mathcal{O}_0^{n+1}}{\left\langle \dfrac{\partial f_t}{\partial z_0}, \dots, \dfrac{\partial f_t}{\partial z_n} \right\rangle}$$

then suddenly the statement that the constancy of the Milnor number implies the constancy of the homotopy-type of the Milnor fibre becomes much more interesting; it becomes a statement that some effectively calculable algebraic data controls some topological/geometric data.

This is the general type of result that we are looking for, and this paper provide the basic results connecting algebraic data with the topological/geometric data given by the hypercohomology of the Milnor fibre with coefficients in an arbitrary complex of sheaves.

For example, let X be an analytic subspace of some open neighbourhood of the origin in $\mathbb{C}^{n+1}$. Assume $\mathbf{0} \in X$. Let $\mathscr{S} = \{S_\alpha\}$ be a complex Whitney stratification of X, and let $\mathbf{F}^\bullet$ be a bounded complex of sheaves of $\mathbb{Z}$-modules which is constructible with respect to $\mathscr{S}$.

Let $f_t : (X, \mathbf{0}) \to (\mathbb{C}, 0)$ be a family of analytic functions such that $\dim_0 \Sigma_{\mathscr{S}} f_t = 0$. Then, Theorem 3.2 says that, if we require

$$(\tilde{\Gamma}_{f_t,L}(S_\alpha)\cdot V(f_t))_0 - (\tilde{\Gamma}^1_{f_t,L}(S_\alpha)\cdot V(L))_0$$

to be independent of t for all $S_\alpha \neq \{\mathbf{0}\}$ (where L is a generic linear form which depends on $\mathscr{S}$, but not on $\mathbf{F}^\bullet$), then the stalk cohomology of the sheaf of vanishing cycles of $\mathbf{F}^\bullet$ along f_t at the origin is independent of t. This is a generalization of the μ-constant result given above.

In analogy with the result of Lê and Saito [25] on Thom's a_f condition for μ-constant families, we fully expect the constancy of

$$(\tilde{\Gamma}^1_{f_t,L}(S_\alpha)\cdot V(f_t))_{\mathbf{0}} - (\tilde{\Gamma}^1_{f_t,L}(S_\alpha)\cdot V(L))_{\mathbf{0}}$$

to imply a_f conditions of some form

However, we are not restricting our future research to studying families of generalized isolated singularities. In [31–33], we develop and explore the Lê cycles of an analytic function, f, on affine space where f is allowed to have a critical locus of arbitrary dimension. Using these Lê cycles, we are able to generalize many results which were previously known only for isolated hypersurface singularities. In particular, we give algebraic data—the Lê numbers—such that the constancy of these data in a family implies the constancy of the cohomology of the Milnor fibres and implies the a_f condition holds.

In [34], we set up the general framework which generalizes the Lê cycles to a collection of data associated to an arbitrary bounded, constructible complex of sheaves. However, the generalization given in [34] is in non-algebraic terms, and so is not terribly useful for calculations.

This present paper supplies the algebraic bridge between our work on affine hypersurface singularities and our abstract generalization to the vanishing cycles of a complex of sheaves along a function on an arbitrary analytic space. A large number of very general results should follow.

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